WORK, POWER & ENERGY

INTRODUCTION :

The term ‘work’ as understood in everyday life has a different meaning in scientific sense. If a coolie is carrying a load on his head and waiting for the arrival of the train, he is not performing any work in the scientific sense. In the present study, we shall have a look into the scientific aspect of this most commonly used term i.e., work.

WORK DONE BY CONSTANT FORCE:

The physical meaning of the term work is entirely different from the meaning attached to it in everyday life. In everyday life, the term ‘work’ is considered to be synonym of ‘labour’, ‘toil’, effort’ etc. In physics, there is a specific way of defining work.

Work is said to be done by a force when the force produces a displacement in the body on which it acts in any direction except perpendicular to the direction of the force.

For work to be done, following two conditions must be fulfilled.

(i) A force must be applied.

(ii) The applied force must produce a displacement in any direction except perpendicular to the direction of the force.

Suppose a force $\vec{F}$ is applied on a body in such a way that the body suffers a displacement $\vec{S}$ in the direction of the force. Then the work done is given by

W = FS

However, the displacement does not always take place in the direction of the force. Suppose a constant force $\vec{F}$, applied on a body, produces a displacement $\vec{S}$ in the body in such a way that $\vec{S}$ is inclined to $\vec{F}$ at an angle θ. Now the work done will be given by the dot product of force and displacement.

W = $\vec{F}.\vec{S}$

Since work is the dot product of two vectors therefore it is a scalar quantity.

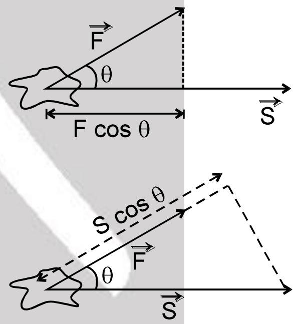
W = FS cos θ or W = (F cos θ)S

$\therefore$ W = component of force in the direction of displacement × magnitudes of displacement.

So work is the product of the component of force in the direction of displacement and the magnitude of the displacement.

Also, W = F (S cos θ)

or work is product of the component of displacement in the direction of the force and the magnitude of the displacement.



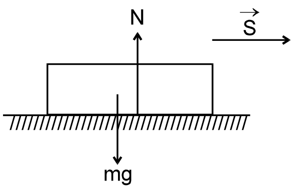
Special Cases :

Case (I) : When θ = 90º, then W = FS cos 90º = 0.

So, work done by a force is zero if the body is displaced in a direction perpendicular to the direction of the force.

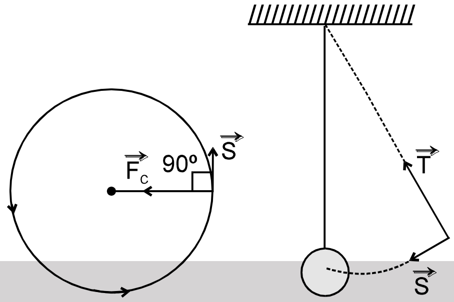
Examples :

1. Consider a body sliding over a horizontal surface. The work done by the force of gravity and the reaction of the surface will be zero. This is because both the force of gravity and the reaction act normally to the displacement.



The same argument can be applied to a man carrying a load on his head and walking on a railway platform.

2. Consider a body moving in a circle with constant speed. At every point of the circular path, the centripetal force and the displacement are mutually perpendicular (Figure). So, the work done by the centripetal force is zero. The same argument can be applied to a satellite moving in a circular orbit. In this case, the gravitational force is always perpendicular to displacement. So, work done by gravitational force is zero.



3. The tension in the string of a simple pendulum is always perpendicular to displacement. (Figure). So, work done by the tension is zero.

Case (II) : When S = 0, then W = 0.

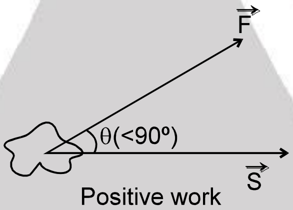
So, work done by a force is zero if the body suffers no displacement on the application of a force.

Example : A person carrying a load on his head and standing at a given place does no work.

Case (III) : When 0º ≤ θ < 90º [Figure], then cos θ is positive. Therefore.

W (= FS cos θ) is positive.

Work done by a force is said to be positive if the applied force has a component in the direction of the displacement.



Positive work

Examples :

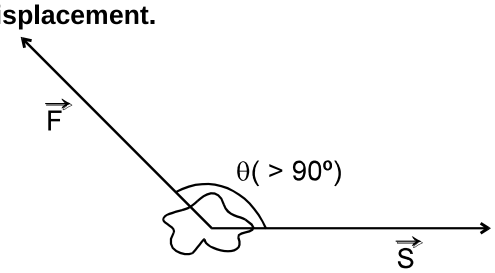
1. When a horse pulls a cart, the applied force and the displacement are in the same direction. So, work done by the horse is positive.

2. When a load is lifted, the lifting force and the displacement act in the same direction. So, work done by the lifting force is positive.

3. When a spring is stretched, both the stretching force and the displacement act in the same direction. So, work done by the stretching force is positive.

Case (IV) : When 90º < θ ≤ 180º (Figure), then cosθ is negative. Therefore W (= FS cos θ) is negative.

Work done by a force is said to be negative if the applied force has component in a direction opposite to that of the displacement.



Negative work

Examples :

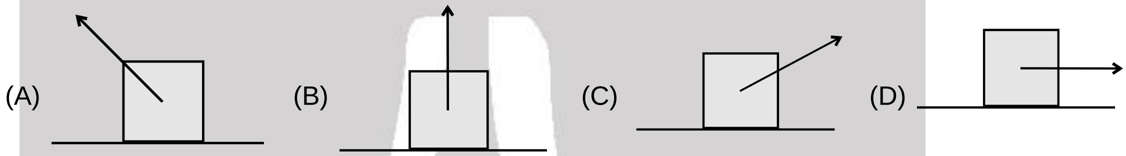
1. When brakes are applied to a moving vehicle, the work done by the braking force is negative. This is because the braking force and the displacement act in opposite directions.

2. When a body is dragged along a rough surface, the work done by the frictional force is negative. This is because the frictional force acts in a direction opposite to that of the displacement.

3. When a body is lifted, the work done by the gravitational force is negative. This is because the gravitational force acts vertically downwards while the displacement is in the vertically upwards direction.

Solved Example

Example 1. Figure shows four situations in which a force acts on a box while the box slides rightward a distance d across a frictionless floor. The magnitudes of the forces are identical, their orientations are as shown. Rank the situations according to the work done on the box during the displacement, from most positive to most negative.



Answer : D, C, B, A

Explanation : In (D) θ = 0º, cos θ = 1 (maximum value). So, work done is maximum.

In (C) θ < 90º, cos θ is positive. Therefore, W is positive.

In (B) θ = 90º, cos θ is zero. W is zero.

In (A) θ is obtuse, cos θ is negative. W is negative.

WORK DONE BY MULTIPLE FORCES :

If several forces act on a particle, then we can replace $\vec{F}$ in equation W = $\vec{F}.\vec{S}$ by the net force $\Sigma\vec{F}$ where

$\Sigma = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + ...$

$\therefore$ $W = [\Sigma\vec{F}] . \vec{S}$ ...(i)

This gives the work done by the net force during a displacement of the particle.

We can rewrite equation (i) as :

$W = \vec{F}_1.\vec{S} + \vec{F}_2\vec{S} + \vec{F}_3\vec{S} + . \quad . \quad . \quad .$

or $W = W_1 + W_2 + W_3 + ...$

So, the work done on the particle is the sum of the individual works done by all the forces acting on the particle.

Important points about work :

1. Work is defined for an interval or displacement. There is no term like instantaneous work similar to instantaneous velocity.
2. For a particular displacement, work done by a force is independent of type of motion i.e. whether it moves with constant velocity, constant acceleration or retardation etc.
3. For a particular displacement work is independent of time. Work will be same for same displacement whether the time taken is small or large.
4. When several forces act, work done by a force for a particular displacement is independent of other forces.
5. A force is independent from reference frame. Its displacement depends on frame so work done by a force is frame dependent therefore work done by a force can be different in different reference frame.
6. Effect of work is change in kinetic energy of the particle or system.
7. Work is done by the source or agent that applies the force.

Units of work :

1. Unit of work :

I. In cgs system, the unit of work is erg.

One erg of work is said to be done when a force of one dyne displaces a body through one centimetre in its own direction.

$\therefore$ 1 erg = 1 dyne × 1 cm = 1g cm s^{-2} × 1 cm = 1 g cm^2 s^{-2}

Note : Erg is also called dyne centimetre.

II. In SI i.e., International System of units, the unit of work is joule (abbreviated as J). It is named after the famous British physicist James Personal Joule (1818 – 1869).

One joule of work is said to be done when a force of one Newton displaces a body through one metre in its own direction.

1 joule = 1 Newton × 1 metre = 1 kg ×1 m/s^2 × 1 m = 1kg m^2 s^{-2}

Note : Another name for joule is Newton metre.

<u>**Relation between joule and erg**</u>

1 joule = 1 Newton × 1 metre ; 1 joule = 10^5 dyne × 10^2 cm = 10^7 dyne cm

1 joule = 10^7 erg ; 1 erg = 10^{-7} joule

DIMENSIONS OF WORK :

[Work] = [Force] [Distance] = [MLT^{-2}] [L] = [ML^2T^{-2}]

Work has one dimension in mass, two dimensions in length and '–2' dimensions in time,

On the basis of dimensional formula, the unit of work is kg m^2 s^{-2}.

Note that 1 kg m^2 s^{-2} = (1 kg m s^{-2}) m = 1 N m = 1 J.

Solved Examples

Example 2. There is an elastic ball and a rigid wall. Ball is thrown towards the wall. The work done by the normal reaction exerted by the wall on the ball is -

(A) +ve (B) – ve (C) zero (D) None of these

Answer : **(C)**

Solution : As the point of application of force does not move, the w.d by normal reaction is zero.

Example 3. Work done by the normal reaction when a person climbs up the stairs is -

(A) +ve (B) – ve (C) zero (D) None of these

Answer : **(C)**

Solution : As the point of application of force does not move, the w.d by normal reaction is zero.

Example 4. Work done by static friction force when a person starts running is ________.

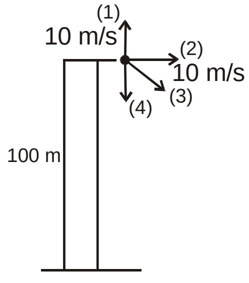
Solution : As the point of application of force does not move, the w.d by static friction is zero.

<u>WORK DONE BY VARIOUS REAL FORCES</u>

Work done by gravity Force.

Solved Example

Example 5. The mass of the particle is 2 kg. It is projected as shown in four different ways with same speed of 10 m/s. Find out the work done by gravity by the time the stone falls on ground.

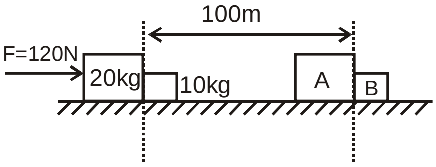


Solution : $W = |\vec{F}||\vec{S}|\cos\theta$ = 2000 J in each case.

Work done by normal reaction.

Solved Example

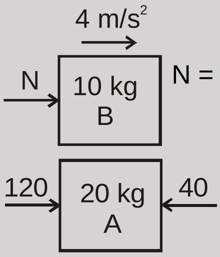
Example 6.



(i) Find work done by force F on A during 100 m displacement.
(ii) Find work done by force F on B during 100 m displacement.
(iii) Find work done by normal reaction on B and A during the given displacement.
(iv) Find out the kinetic energy of block A & B finally.

Solution : (i) $(W_F)_{on\ A} = F\Delta S \cos\theta$
$= 120 \times 100 \times \cos 0° = 12000$ J

(ii) $(W_F)_{on\ B} = 0$
F does not act on B

(iii)


N = 10 × 4 = 40 N

$(W_N)_{on\ B} = 40 \times 100 \times \cos 0° = 4000$ J
$(W_N)_{on\ A} = 40 \times 100 \times \cos 180° = -4000$ J

(iv) $v^2 = u^2 + 2as$ $\quad$ $u = 0$

$\therefore\ v^2 = 2 \times 4 \times 100 \Rightarrow v = 20\sqrt{2}$ m/s

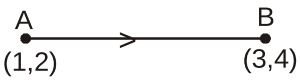
$\therefore\ KE_A = \frac{1}{2} \times 20 \times 800 = 8000$ J

$KE_B = \frac{1}{2} \times 10 \times 800 = 4000$ J

W.D. by normal reaction on system of A & B is zero. i.e. w.d. by internal reaction on a rigid system is zero.

Example 7. A particle is displaced from point A (1, 2) to B(3, 4) by applying force $\vec{F} = 2\hat{i} + 3\hat{j}$. Find the work done by $\vec{F}$ to move the particle from point A to B.

Solution : $W = \vec{F} \cdot \Delta\vec{s}$



$\Delta S = (3-1)\hat{i} + (4-2)\hat{j} = (2\hat{i} + 3\hat{j}) \cdot (2\hat{i} + 2\hat{j}) = 2 \times 2 + 3 \times 2 = 10$ units

ENERGY :

Definition: Energy is defined as internal capacity of doing work. When we say that a body has energy we mean that it can do work.

Energy appears in many forms such as mechanical, electrical, chemical, thermal (heat), optical (light), acoustical (sound), molecular, atomic, nuclear etc., and can change from one form to the other.

KINETIC ENERGY :

Definition : Kinetic energy is the internal capacity of doing work of the object by virtue of its motion. Kinetic energy is a scalar property that is associated with state of motion of an object. An aero-plane in straight and level flight has kinetic energy of translation and a rotating wheel on a machine has kinetic energy of rotation. If a particle of mass m is moving with speed 'v' much less than the speed of the light than the kinetic energy 'K' is given by $K = \frac{1}{2}mv^2$

Important Points for K.E.

1. As mass m and v^2 ($\vec{v}.\vec{v}$) are always positive, kinetic energy is always positive scalar i.e, kinetic energy can never be negative.

2. The kinetic energy depends on the frame of reference,

$K = \frac{p^2}{2m}$ and $P = \sqrt{2mK}$; P = linear momentum

The speed v may be acquired by the body in any manner. The kinetic energy of a group of particles or bodies is the sum of the kinetic energies of the individual particles. Consider a system consisting of n particles of masses m_1, m_2, ..., m_n. Let $\vec{v_1}$, $\vec{v_2}$, ... , $\vec{v_n}$ be their respective velocities. Then, the total kinetic energy E_k of the system is given by

$$E_k = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 + ... + \frac{1}{2}m_nv_n^2$$

If m is measured in gram and v in cm s^{-1}, then the kinetic energy is measured in erg. If m is measured in kilogram and v in m s^{-1}, then the kinetic energy is measured in joule. It may be noted that the units of kinetic energy are the same as those of work. Infect, this is true of all forms of energy since they are inter-convertible.

Typical kinetic energies (K) :

S.No.	Object	Mass (kg)	Speed (m s^{-1})	K(J)
1	Air molecule	$\approx 10^{-26}$	500	$\approx 10^{-21}$
2	Rain drop at terminal speed	3.5×10^{-5}	9	1.4×10^{-3}
3	Stone dropped from 10 m	1	14	10^2
4	Bullet	5×10^{-5}	200	10^3
5	Running athlete	70	10	3.5×10^3
6	Car	2000	25	6.3×10^5

RELATION BETWEEN MOMENTUM AND KINETIC ENERGY :

Consider a body of mass m moving with velocity v. Linear momentum of the body, p = mv

Kinetic energy of the body, $E_k = \frac{1}{2}mv^2$

$E_k = \frac{1}{2m}(m^2v^2)$ or $E_k = \frac{p^2}{2m}$ or $p = \sqrt{2mE_k}$

Solved Example

Example 8. The kinetic energy of a body is increased by 21%. What is the percentage increase in the magnitude of linear momentum of the body?

Solution : $E_{k2} = \frac{121}{100}E_{k1}$ or $\frac{1}{2}m\,v_2^2 = \frac{121}{100}\frac{1}{2}mv_1^2$ or $v_2 = \frac{11}{10}v_1$

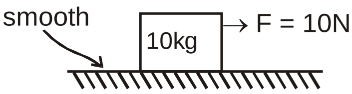
or $mv_2 = \frac{11}{10}mv_1$ or $p_2 = \frac{11}{10}p_1$

or $\frac{p_2}{p_1} - 1 = \frac{11}{10} - 1 = \frac{1}{10}$

or $\frac{p_2 - p_1}{p_1} \times 100 = \frac{1}{10} \times 100 = 10$

So, the percentage increase in the magnitude of linear momentum is 10%.

Example 9.



Force shown acts for 2 seconds. Find out w.d. by force F on 10 kg in 3 seconds.

Solution : $w = \vec{F}.\Delta\vec{S} \Rightarrow w = \vec{F}.\Delta\vec{S}.\cos 0° \Rightarrow w = 10\,\Delta\vec{S}$

Now $10 = 10\,a \quad \therefore \quad a = 1\ m/s^2 \Rightarrow S = \frac{1}{2}at^2 = \frac{1}{2} \times 1 \times 2^2 = 2\ m$

$w = 10 \times 2 = 20\ J$

Example 10. Find Kinetic energy after 2 seconds.

Solution : $V = 0 + at \Rightarrow V = 1 \times 2 = 2m/s$

$\therefore \quad KE = \frac{1}{2} \times 10 \times 2^2 = 20\ J.$

WORK DONE BY A VARIABLE FORCE :

When the magnitude and direction of a force vary in three dimensions, it can be expressed as a function of the position. For a variable force work is calculated for infinitely small displacement and for this displacement force is assumed to be constant

$$\mathbf{dW = \vec{F}.d\vec{s}}$$

The total work done will be sum of infinitely small work

$$W_{A \to B} = \int_A^B \vec{F} \cdot d\vec{s} = \int_A^B (F\cos\theta)ds$$

In terms of rectangular components,

$$\vec{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k} \quad \Rightarrow \quad d\vec{s} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$$W_{A\to B} = \int_{x_A}^{x_B} F_x dx + \int_{y_A}^{y_B} F_y dy + \int_{z_A}^{z_B} F_z dz$$

Solved Example

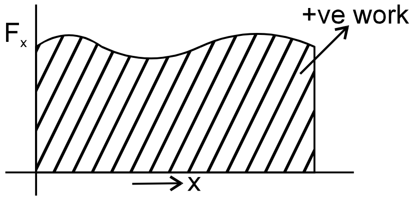
Example 11. An object is displaced from position vector $\vec{r}_1 = (2\hat{i} + 3\hat{j})m$ to $\vec{r}_2 = (4\hat{i} + 6\hat{j})m$ under the action of a force $\vec{F} = (3x^2\hat{i} + 2y\hat{j})N$. Find the work done by this force.

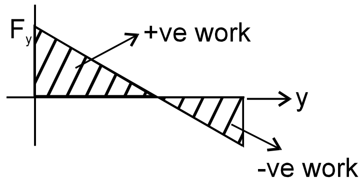
Solution : $W = \int_{\vec{r}_i}^{\vec{r}_f} \vec{F} \bullet d\vec{r} = \int_{\vec{r}_1}^{\vec{r}_2} (3x^2\hat{i} + 2y\hat{j}) \bullet (dx\hat{i} + dy\hat{j} + dz\hat{k})$

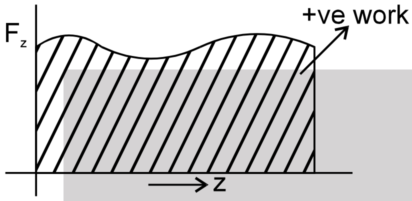
$= \int_{\vec{r}_1}^{\vec{r}_2} (3x^2dx + 2ydy) = [x^3 + y^2]_{(2,\ 3)}^{(4,\ 6)} = 83\ J$ **Ans.**

AREA UNDER FORCE DISPLACEMENT CURVE :

Graphically area under the force-displacement is the work done







The work done can be positive or negative as per the area above the x-axis or below the x-axis respectively.

Solved Example

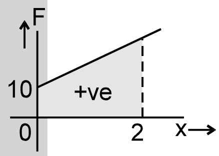
Example 12. A force F = 0.5x + 10 acts on a particle. Here F is in Newton and x is in metre. Calculate the work done by the force during the displacement of the particle from x = 0 to x = 2 metre.

Solution : Small amount of work done dW in giving a small displacement $d\vec{x}$ is given by

$dW = \vec{F}.d\vec{x}$

or $dW = Fdx \cos 0^o$

or $dW = Fdx$ $[\because \cos 0^o = 1]$



Total work done, $W = \int_{x=0}^{x=2} Fdx = \int_{x=0}^{x=2} (0.5\,x + 10)dx$

$$= \int_{x=0}^{x=2} 0.5x \quad dx + \int_{x=0}^{x=2} 10\,dx = 0.5 \left|\frac{x^2}{2}\right|_{x=0}^{x=2} + 10\left|\, x \,\right|_{x=0}^{x=2}$$

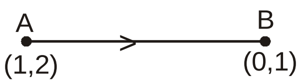
$$= \frac{0.5}{2}\,[2^2 - 0^2] + 10[2 - 0] = (1 + 20) = 21\text{ J}$$

Work done by Variable Force $W = \int dW = \int \vec{F}\cdot\vec{ds}$

Solved Example

Example 13. An object is displaced from point A(1, 2) to B(0, 1) by applying force $\vec{F} = x\hat{i} + 2y\hat{j}$. Find out work done by $\vec{F}$ to move the object from point A to B.

Solution : $dW = \vec{F}.\vec{ds}$



$dW = (x\hat{i} + 2y\hat{j})\,(dx\,\hat{i} + dy\,\hat{j})$

$dW = \int_1^0 x\,dx + \int_2^1 2y\,dy$

$\therefore W = -3.5$ J

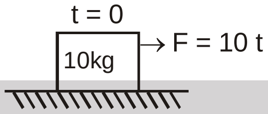
Example 14. The linear momentum of a body is increased by 10%. What is the percentage change in its kinetic energy?

Solution : Percentage increase in kinetic energy = 21%]

$$\left[\ \text{Hint.}mv_2 = \frac{110}{100}mv_1,\ \ v_2 = \frac{11}{10}v_1, \frac{E_2}{E_1} = \left(\frac{11}{10}\right)^2 = \frac{121}{100}\right.$$

$$\left.\text{Percentage increase in kinetic energy} = \frac{E_2 - E_1}{E_1} \times 100\right] = 21\%$$

Example 15. A time dependent force F = 10 t is applied on 10 kg block as shown in figure.



Find out the work done by F in 2 seconds.

Solution : $dW = \vec{F}.d\vec{s}$

dW = 10 t. dx

dW = 10 t v dt .... (1) dx = vdt

also $10 \times \frac{dv}{dt} = 10\ t$

$\therefore\ \int_0^v dv = \int_0^t t\,dt \Rightarrow v = \frac{t^2}{2}$ .... (2)

from (1) & (2)

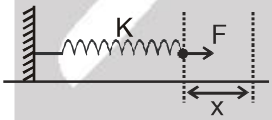
$dW = 10\ t\ .\left(\frac{t^2}{2}\right) dt\ ;\ dW = 5t^3dt\ ;\ W = \frac{5}{4}\left[t^4\right]_0^2 = 20\ J$

Aliter : $\Delta K.E. = \frac{1}{2} \times 10\ (2^2 - 0) = 0$

WORK DONE BY SPRING FORCE

Solved Example

Example 16.



Initially spring is relaxed. A person starts pulling the spring by applying a variable force F. Find out the work done by F to stretch it slowly to a distance by x.

Solution : $\int dW = \int F \bullet ds = \int_0^x Kxdx \Rightarrow \quad W = \left(\frac{Kx^2}{2}\right)_0^x = \frac{Kx^2}{2}$

Example 17. In the above example

(i) Where has the work gone ?

(ii) Work done by spring on wall is zero. Why?

(iii) Work done by spring force on man is _______ .

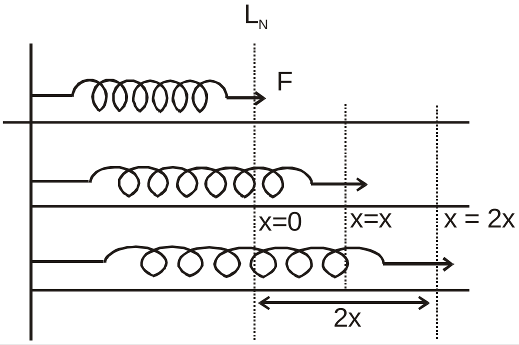
Solution : (i) It is stored in the form of potential energy in spring.

(ii) Zero, as displacement is zero.

(iii) $-\frac{1}{2}Kx^2$

Example 18. Find out work done by applied force to slowly extend the spring from x to 2x.

Solution : Initially the spring is extended by x



$W = \vec{F}.d\vec{s}$

$W = \int_x^{2x} Kx.dx$

$W = \left[\frac{Kx^2}{2}\right]_x^{2x} = \frac{3}{2}Kx^2$

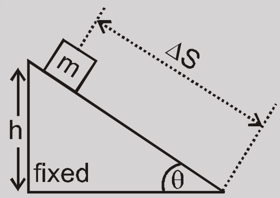
It can also found by difference of PE.

i.e., $U_f = \frac{1}{2}K(2x)^2 = 2kx^2 \Rightarrow U_i = \frac{1}{2}kx^2 \Rightarrow U_f - U_i = \frac{3}{2}kx^2$

WORK DONE BY OTHER CONSTANT FORCES

Solved Example

Example 19. A block of mass m is released from top of a smooth fixed inclined plane of inclination θ.



Find out work done by normal reaction & gravity during the time block comes to bottom.

Solution : $W_N = 0$ as $F \perp \Delta S$

$W_g = \vec{F}.\Delta\vec{S} = mg\,.\,\Delta S\,.\,\cos(90 - \theta) = mg\,\Delta S \sin\theta = mgh$

Example 20. Find out the speed of the block at the bottom and its kinetic energy.

Solution : $V^2 = u^2 + 2as$

$V^2 = 0 + 2(g\sin\theta)\,\frac{h}{\sin\theta}$

$\Rightarrow\ V^2 = 2gh$

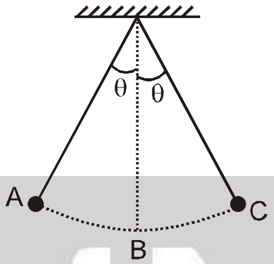
$\Rightarrow\ V = \sqrt{2gh}$

$KE = \frac{1}{2}mv^2 = mgh$

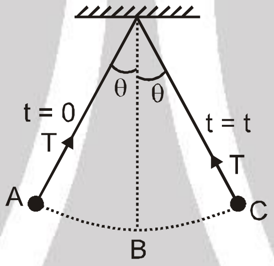
WORK DONE BY TENSION

Solved Example

Example 21. A bob of pendulum is released at rest from extreme position as shown in figure. Find work done by tension from A to B, B to C and C to A.



Solution : Zero because $F_T \perp dS$ at all time.



Example 22. In the above question find out work done by gravity from A to B and B to C.

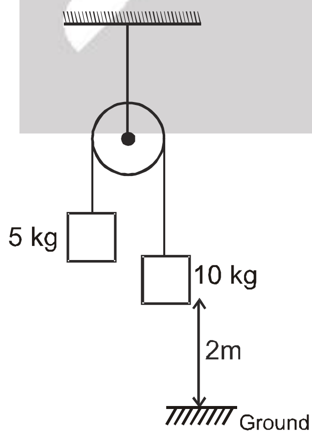
Solution : $W_g = \vec{F} . \Delta\vec{S}$

$= mg\ \Delta S \cos\theta$

$W_g = mg\ (\ell - \ell \cos\theta)$ for A to B

$W_g = -\ mg\ (\ell - \ell \cos\theta)$ for B to C

Example 23.



The system is released from rest. When 10 kg block reaches at ground then find :
(i) Work done by gravity on 10 kg (ii) Work done by gravity on 5 kg
(iii) Work done by tension on 10 kg (iv) Work done by tension on 5 kg.

Solution : (i) $(W_g)_{10\text{ kg}} = 10\text{ g} \times 2 = 200$ J

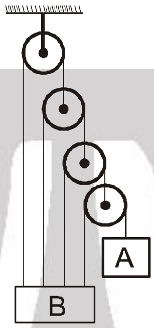
(ii) $(W_g)_{5kg} = 5\text{ g} \times 2 \times \cos 180° = -100\text{ J}$

(iii) $(W_T)_{10\text{ kg}} = \dfrac{200}{3} \times 2 \times \cos 180° = \dfrac{-400}{3}\text{ J}$

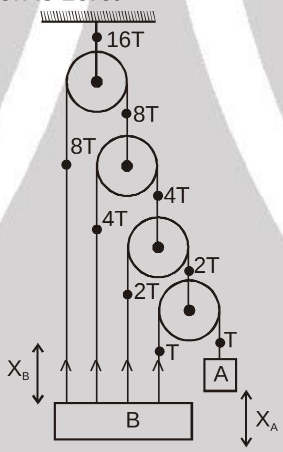
(iv) $(W_T)_{5\text{ kg}} = \dfrac{200}{3} \times 2 \times \cos 0° = \dfrac{400}{3}\text{ J}$

Net w.d. by tension is zero. Work done by internal tension i.e. (tension acting within system) on the system is always zero if the length remains constant.

Example 24. The velocity block A of the system shown in figure is V_A at any instant. Calculate velocity of bock B at that instant.



Solution : Work done by internal tension is zero.

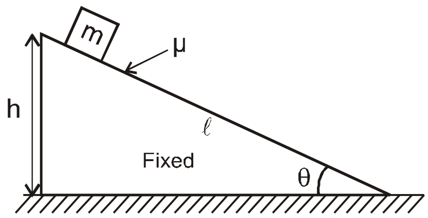


$\therefore \quad 15\text{ T} \times X_B - \text{T} \times X_A = 0$

$X_A = 15\, X_B$

$\therefore \quad V_A = 15\, V_B$

Example 25. A block of mass m is released from top of an incline plane of inclination θ. The coefficient of friction between the block and incline surface is μ ($\mu < \tan\theta$). Find work done by normal reaction, gravity & friction, when block moves from top to the bottom.



Solution : $W_N = 0 \qquad \therefore\ F_N \perp \Delta S$

$W_g = mg\,\ell \sin\theta$

$W_f = -\mu mg \cos\theta.\ell$

Example 26. What is kinetic energy of block of mass m at bottom in above problem ?

Solution : $V^2 = u^2 + 2as$

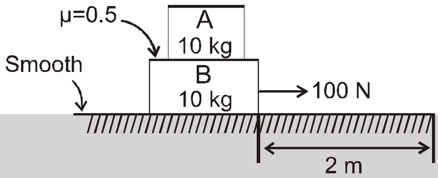
$V^2 = 2(g\sin\theta - \mu g\cos\theta)(\ell)$

$\therefore\quad KE = m\,2\dfrac{1}{2}\,(g\sin\theta - \mu g\cos\theta)\,\ell = mg\,\ell\,(\sin\theta - \mu\cos\theta)$

WORK DONE BY FRICTION

Solved Example

Example 27. In the given figure



(i) Find work done by applied force during displacement 2m.
(ii) Find work done by frictional force on B by A during the displacement.

Solution : (i) $100 \times 2 \times \cos 0° = 200$ J

(ii) $f_{smax} = \mu mg = 0.5 \times 10\,g = 50$ N

Assuming they move together.

$100 = 20\,a \Rightarrow a = 5\ m/s^2$

Check Friction on A ; $f = 10 \times 5 = 50$ N

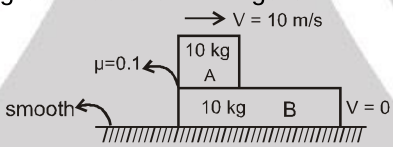
$f_{reqd} = f_{available}$ $\therefore$ They move together

Hence $(W_f)_{on\ B} = -100$ J

$(W_f)_{on\ A} = 100$J. Net zero

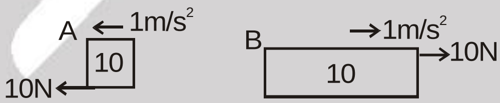
i.e. w.d. by internal static friction is zero.

Example 28. A block of mass 10 kg is projected with speed 10 m/s on the surface of the plank of mass 10 kg, kept on smooth ground as shown in figure.



(i) Find out the velocity of two blocks when frictional force stops acting.
(ii) Find out displacement of A & B till velocity becomes equal.

Solution : (i)



$V_A = 10 - 1t \quad\Rightarrow\quad V_B = 1t$

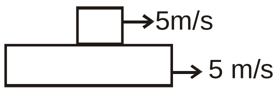
Frictional force stops acting when

$V_A = V_B \quad\Rightarrow\quad 10 - t = t$

$10 = 2t \quad\Rightarrow\quad t = 5$ sec.

$V_B = V_A = 5$m/s

Situation becomes



(ii) $S_A = 10 \times 5 - \dfrac{1}{2} \times 1 \times 5^2 = 37.5$ m

$S_B = \dfrac{1}{2} \times 1 \times 5^2 = 12.5$ m

Example 29. In the above question find work done by kinetic friction on A & B.

Solution : $(W_{KF})_{on}\ A = 10 \times 37.5\cos 180° = -375$ J

$(W_{KF})_{on}$ B = 10 × 12.5 cos 0 = 125 J

work done by KF on system of A & B = – 375 + 125 = – 250 J

Work done by KF on a system is always negative.

Heat generated = – (W_{KF}) on system

(W_{KF}) on system = – $(f_K \times S_{relative})$ = – 10 × 25 = – 250 J

True / False :

Example 30. Work done by kinetic friction on a body is never zero.

Answer : False

Example 31. Work done by kinetic friction on a system is always negative.

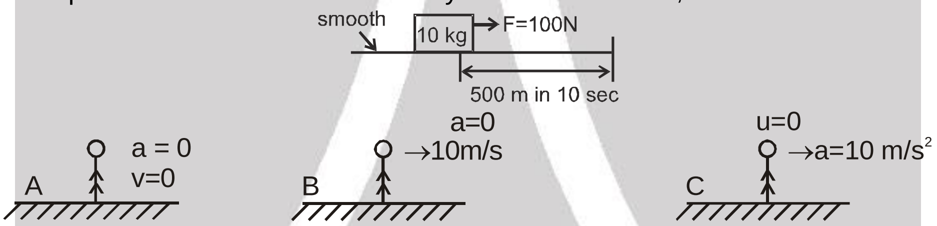
Answer : True

WORK DONE BY PSEUDO FORCE

Kinetic Energy of a body frame dependent as velocity is a frame dependent quantity. Therefore pseudo force work has to be considered.

Solved Example

Example 32. A block of mass 10 kg is pulled by force F = 100 N. It covers a distance 500 m in 10 sec. From initial point. This motion is observed by three observers A, B and C as shown in figure.



Find out work done by the force F in 10 seconds as observed by A, B & C.

Solution : $(W_F)_{\text{on block w.r.t A}}$ = 100 × 500 J = 50,000 J

$(W_F)_{\text{on block w.r.t B}}$ = 100 [500 – 10 × 10] = 40,000 J

$(W_F)_{\text{on block w.r.t C}}$ = 100 [500 – 500] = 0

WORK DONE BY INTERNAL FORCE

$F_{AB} = - F_{BA}$ i.e. sum of internal forces is zero.

But it is not necessary that work done by internal force is zero. There must be some deformation or reformation between the system to do internal work. In case of a rigid body work done by internal force is zero.

Work-Energy Theorem :

According to work-energy theorem, the work done by all the forces on a particle is equal to the change in its kinetic energy.

$W_C + W_{NC} + W_{PS} = \Delta K$

Where, W_C is the work done by all the conservative forces.

W_{NC} is the work done by all non-conservative forces.

W_{PS} is the work done by all psuedo forces.

Modified Form of Work-Energy Theorem :

We know that conservative forces are associated with the concept of potential energy, that is

$W_C = - \Delta U$

So, Work-Energy theorem may be modified as

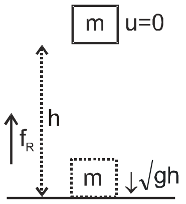
$W_{NC} + W_{PS} = \Delta K + \Delta U$

$W_{NC} + W_{PS} = \Delta E$

Solved Examples

Example 33. A body of mass m when released from rest from a height h, hits the ground with speed $\sqrt{gh}$. Find work done by resistive force.

Solution : Identify initial and final state and identify all forces.



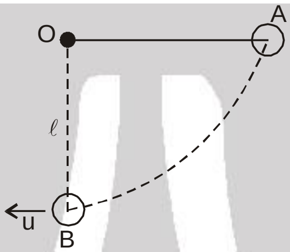
$$W_g + W_{air\ res.} + W_{int\ force} = \Delta K$$

$$mgh + W_{air\ res} + 0 = \frac{1}{2} m \left(\sqrt{gh}\right)^2 - 0$$

$$\Rightarrow \quad W_{air\ res.} = -\frac{mgh}{2}$$

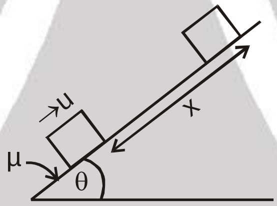
Example 34. The bob of a simple pendulum of length l is released when the string is horizontal. Find its speed at the bottom.

Solution : $W_g + W_T = \Delta K$



$$mg\ell + 0 = \frac{1}{2} mu^2 - 0 \ ; \ \ u = \sqrt{2g\ell}$$

Example 35. A block is given a speed u up the inclined plane as shown.

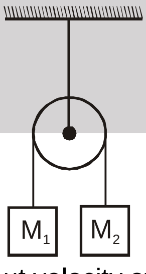


Using work energy theorem find out x when the block stops moving.

Solution : $W_g + W_f + W_N = \Delta K$

$$-\ mg\ x \sin\theta - \mu\ mgx \cos\theta + 0 = 0 - \frac{1}{2} mu^2 \qquad \Rightarrow \qquad x = \frac{u^2}{2g(\sin\theta + \mu\cos\theta)}$$

Example 36. The masses M_1 and M_2 ($M_2 > M_1$) are released from rest.



Using work energy theorem find out velocity of the blocks when they move a distance x.

Solution : $(W_{all\ F})_{system} = (\Delta K)_{system}$

$(W_g)_{sys} + (W_T)_{sys} = (\Delta K)_{sys}$ as $(W_T)_{sys} = 0$

$$M_2gx - M_1gx = \frac{1}{2} (M_1 + M_2)V^2 - 0 \qquad .... (1)$$

$$V = \sqrt{\frac{2(M_2 - M_1)gx}{M_1 + M_2}}$$

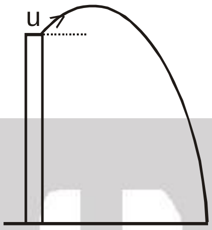
Example 37. In the above question find out acceleration of blocks.

Solution : $(M_2g - M_1g) = \frac{1}{2}(M_1 + M_2)\, 2v\frac{dv}{dx}$ [Differentiating equation (1) above]

$$\Rightarrow \left(\frac{M_2 - M_1}{M_1 + M_2}\right)g = v\frac{dv}{dx} = a$$

Example 38. A stone is projected with initial velocity u from a building of height h. After some time the stone falls on ground. Find out speed with it strikes the ground.

Solution : $W_{\text{all forces}} = \Delta K$



$W_g = \Delta K$

$mgh = \frac{1}{2}mv^2 - \frac{1}{2}mu^2$

$v = \sqrt{u^2 + 2gh}$

Power :

Power is defined as the time rate of doing work.

When the time taken to complete a given amount of work is important, we measure the power of the agent of doing work.

The average power ($\bar{P}$ or p_{av}) delivered by an agent is given by

$\bar{P}$ or $p_{av} = \frac{W}{t}$ where W is the amount of work done in time t.

Power is the ratio of two scalars- work and time. So, power is a scalar quantity. If time taken to complete a given amount of work is more, then power is less. For a short duration dt, if P is the power delivered during this duration, then

$$P = \frac{\vec{F}\cdot d\vec{S}}{dt} = \vec{F}\cdot\frac{d\vec{S}}{dt} = \vec{F}\cdot\vec{v}$$

This is instantaneous power. It may be +ve, –ve or zero.

By definition of dot product,

$$P = Fv\cos\theta$$

where θ is the smaller angle between $\vec{F}$ and $\vec{v}$.

This P is called as instantaneous power if dt is very small.

Solved Example

Example 39. A block moves in uniform circular motion because a cord tied to the block is anchored at the centre of a circle. Is the power of the force exerted on the block by the cord is positive, negative or zero?

Answer : Zero

Explanation. $\vec{F}$ and $\vec{v}$ are perpendicular.

$\therefore$ Power $= \vec{F}\cdot\vec{v} = Fv\cos 90^o =$ Zero.

Unit of Power :

A unit power is the power of an agent which does unit work in unit time.

The power of an agent is said to be one watt if it does one joule of work in one second.

1 watt = 1 joule/second = 10^7 erg/second

Also, $1 \text{ watt} = \dfrac{1\text{newton} \times 1\text{metre}}{1\text{second}} = 1\text{Nms}^{-1}$.

Dimensional formula of power

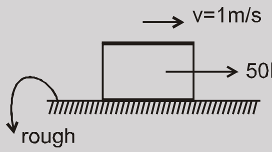
$$[\text{Power}] = \frac{[\text{Work}]}{[\text{Time}]} = \frac{[ML^2T^{-2}]}{[T]} = [ML^2T^{-3}]$$

Power has 1 dimension in mass, 2 dimensions in length and – 3 dimensions in time.

S.No.	Human Activity	Power (W)
1	Heart beat	1.2
2	Sleeping	83
3	Sitting	120
4	Riding in a car	140
5	Walking (4.8 km h^{-1})	265
6	Cycling (15 km h^{-1})	410
7	Playing Tennis	440
8	Swimming (breaststroke, 1.6 km h^{-1})	475
9	Skating	535
10	Climbing Stairs (116 steps min^{-1})	685
11	Cycling (21.3 km h^{-1})	700
12	Playing Basketball	800
13	Tube light	40
14	Fan	60

Solved Example

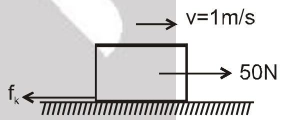
Example 40. A block moves with constant velocity 1 m/s under the action of horizontal force 50 N on a horizontal surface. What is the power of external force and friction?



Solution : Since a = 0 i.e. f_k = 50 N

$P_{ext} = 50 \times 1 = 50$ W

$P_f = -50 \times 1 = -50$ W



Power is also the rate at which energy is supplied.

Net power = $P_1 + P_2 + P_3$ ....

$$P_{net} = \frac{dW_1}{dt} + \frac{dW_2}{dt} \text{......} \quad \Rightarrow \quad P_{net} = \left(\frac{dW_1 + dW_2 + \text{..........}}{dt}\right)$$

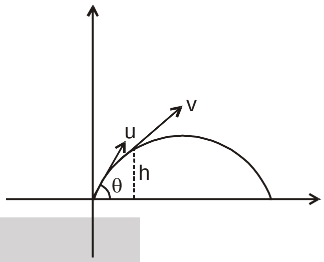
$$P_{net} = \frac{dK}{dt} \qquad \because W_{all} = \Delta K$$

$\therefore$ Rate of change of kinetic energy is also power.

Solved Example

Example 41. A stone is projected with velocity at an angle θ with horizontal. Find out

(i) Average power of the gravity during time t.
(ii) Instantaneous power due to gravitational force at time t where t is time of flight.
(iii) When is average power zero ?
(iv) When is P_{inst} zero ?
(v) When is P_{inst} negative ?
(vi) When is P_{inst} positive ?



Solution : (i) $<P> = \frac{w}{T} = -\frac{mgh}{t} = -\frac{mg\left[u\sin\theta t - \frac{1}{2}gt^2\right]}{t}$

$<P> = mg\left[\frac{gt}{2} - u\sin\theta\right]$

(ii) Instantaneous power

$P = \vec{F}\cdot\vec{v} = (-mg\,\hat{j})\,[u\cos\theta\ \hat{i} + (u\sin\theta - gt))\,\hat{j}]$

$= -mg(u\sin\theta - gt)$

(iii) $\frac{gt}{2} = u\sin\theta \Rightarrow \quad t = \frac{2u\sin\theta}{g}$, i.e. time of flight.

(iv) When $\vec{F}$ & $\vec{V}$ are $\perp$ i.e. at $t = \frac{u\sin\theta}{g}$ which is at the highest point.

(v) From base to highest point.
(vi) From highest point to base.

POTENTIAL ENERGY

Energy : It is the internal capacity to do work.

Kinetic Energy : It is internal capacity to do work by virtue of relative motion.

Potential Energy : It is the internal capacity to do work by virtue of relative position.
Example : Gravitational Potential Energy , Spring PE etc.

Potential Energy
Definition:
Potential energy is the internal capacity of doing work of a system by virtue of its configuration.
In case of conservative force (field) potential energy is equal to negative of work done by the conservative force in shifting the body from some reference position to given position.
Therefore, in case of conservative force

$\int_{U_1}^{U_2} dU = -\int_{r_1}^{r_2} \vec{F}\cdot d\vec{r}$ i.e. $U_2 - U_1 = -\int_{r_1}^{r_2} \vec{F}\cdot d\vec{r} = -W$

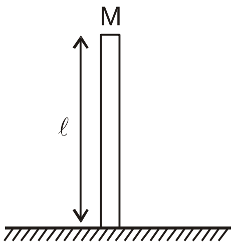
Whenever and wherever possible, we take the reference point at ∞ and assume potential energy to be zero there, i.e., If we take $r_1 = \infty$ and $U_1 = 0$ then

$U = -\int_{\infty}^{r} \vec{F}\cdot d\vec{r} = -W$

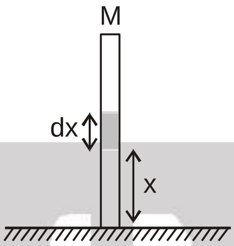
(a) Gravitation Potential Energy :
U = mgh for a particle at a height h above reference level.

Solved Example

Example 42. Calculate potential energy of a uniform vertical rod of mass M and length ℓ.



Solution : dU = (dm) gx



$$\int_0^U dU = \int_0^\ell \left(\frac{M}{\ell}dx\right)gx$$

$$U = \frac{Mg\ell}{2}$$

(b) Spring potential energy :

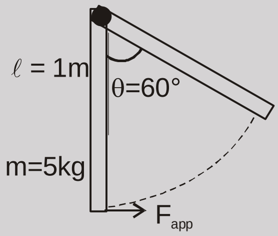
$U = \frac{1}{2}Kx^2$

Where x is change in length from its natural length.

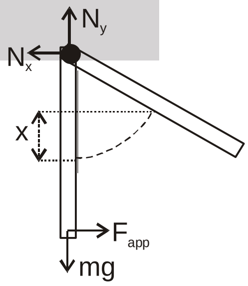
Note : Gravitational potential energy can be +ve , –ve or zero but spring potential energy will always be +ve.

Solved Example

Example 43. In the given figure, a uniform rod of mass m and length l is hinged at one end. Find the work done by applied force in slowly bringing the rod to the inclined position.



Solution : $W_{ALL} = \Delta K$ by work energy theorem



$x = \frac{\ell}{2} - \frac{\ell}{2}\cos 60° = \frac{1}{4}m = 0.25$ m

$W_{N_x} + W_{N_y} + W_g + W_{Fapp} = \Delta K \quad (W_{N_x} = W_{N_y} = 0)$

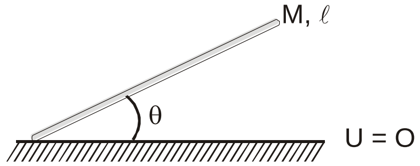
$\therefore$ 0 – mg(0.25) + W_{Fapp} = 0 – 0

$\because$ $\Delta K = 0$ as slowly brought

$\therefore$ W_{Fapp} = 5 × 9.8 × 0.25

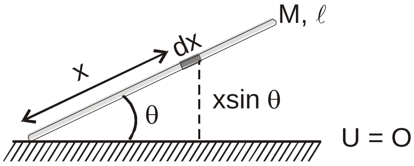
It can be seen that W_{Fapp} = mgh = 5 × 9.8 × 0.25 J.

Example 44. A uniform rod of mass M and length ℓ is held in position shown in the figure. Find the potential energy of the rod.



Solution : dU = dm.g.h

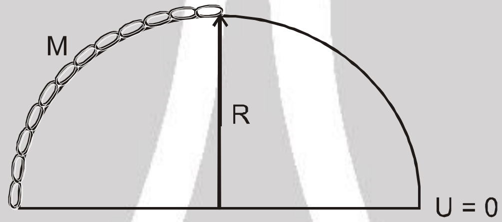
$\int dU = \int_0^{\ell} \frac{M}{\ell}\, dx \, .\, g \, .\, x \sin\theta$



$\therefore \quad U = \frac{Mg\sin\theta\ell}{2}$

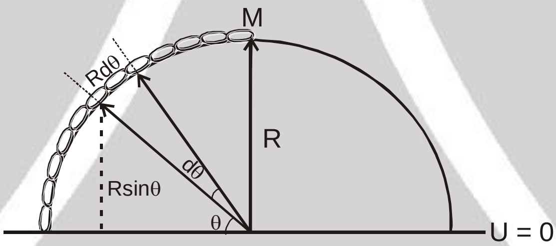
Note that centre of mass of the rod is at height $\left(\frac{\ell}{2}\sin\theta\right)$ from the ground.

Example 45.



A chain of mass M is kept on a hemisphere as shown. Find potential energy of the chain.

Solution :



We know that $\frac{\text{arc}}{\text{Radius}} = \theta$

$\therefore \quad$ elemental length = $Rd\theta$

$\therefore \quad dm = \frac{M}{\pi/2}d\theta = \frac{2M}{\pi}d\theta$

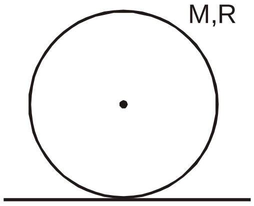
Now $dU = dmgh = \left(\frac{2M}{\pi}d\theta\right)(g)(R\sin\theta)$

$\therefore \quad \int_0^U dU = \frac{2M}{\pi}Rg\int_0^{\pi/2}\sin\theta\, d\theta$

$U = \frac{2MgR}{\pi}\left(-\cos\theta\right)_0^{\pi/2}$

$\Rightarrow \quad U = Mg\left(\frac{2R}{\pi}\right)$ [Note that $\left(\frac{2R}{\pi}\right)$ is the height of COM]

Example 46. A uniform solid sphere of mass M and radius R is kept on the horizontal surface.



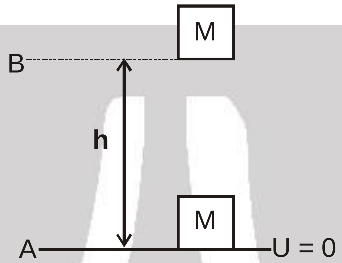
Find potential energy of the solid sphere.

Solution : $U = Mgh_{cm}$

for symmetrical body COM is the geometrical centre of the body.

$\therefore$ $U = MgR$

Example 47. Find work done by gravity in moving the block



(i) from A to B (ii) from B to A (iii) Calculate $U_A - U_B$ (iv) Calculate $U_B - U_A$

Solution : $(W_g)_{A\text{ to }B} = -mgh$

$(W_g)_{B\text{ to }A} = mgh$

$U_A - U_B = -mgh$

$U_B - U_A = mgh$

It can be said

$\mathbf{W_g = -\Delta U}$

Work done by gravity is the -ve of the change in PE.

i.e. $W_g = -[U_f - U_i]$ $\Rightarrow$ $W_g = U_i - U_f$

Important Points for P.E. :

1. Potential energy can be defined only for conservative forces. It has no relevance for non-conservative forces.

2. Potential energy can be positive or negative, depending upon choice of frame of reference.

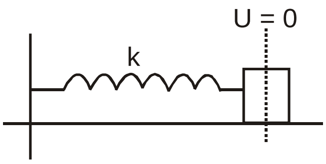
3. Potential energy depends on frame of reference but change in potential energy is independent of reference frame.

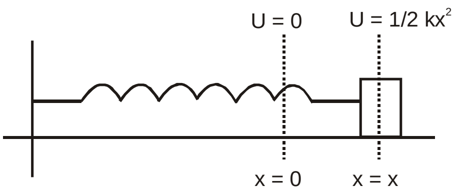
4. Potential energy should be considered to be a property of the entire system, rather than assigning it to any specific particle.

5. It is a function of position and does not depend on the path.

Work done by spring force

As above $W_{SP} = -\Delta U$





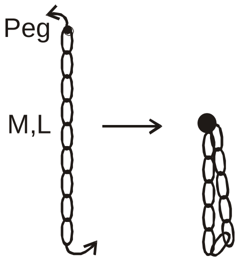
$\therefore$ $W_{SPF} = -\Delta U$

$W_{SPF} = U_i - U_F$

$W_{SPF} = 0 - \frac{1}{2}kx^2 = -\frac{1}{2}kx^2$

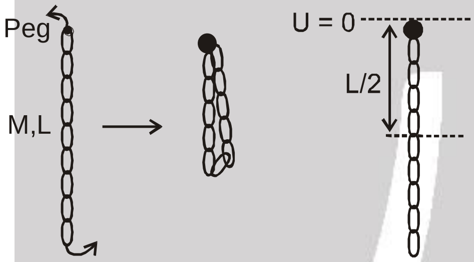
Solved Example

Example 48.

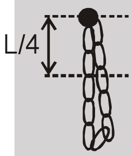


Find out work done by external agent to slowly hang the lower end of the chain to the peg.

Solution :



Initially $U_i = -Mg\frac{L}{2}$



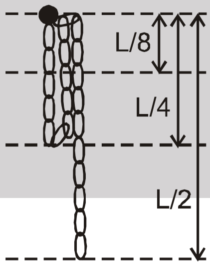
$U_f = -Mg\ \frac{L}{4}$

$\therefore\ W_g = U_i - U_f = (-Mg\frac{L}{2}) - (-Mg\frac{L}{4}) = -Mg\frac{L}{4}$

Using work energy theorem, $W_g + W_{ext} = \Delta K = 0 \quad \Rightarrow \quad W_{ext} = -W_g = Mg\frac{L}{4}$

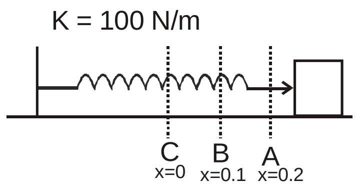
Example 49. In above example find out the work done by external agent to slowly hang the middle link to peg.

Solution : $U_i = -Mg\frac{L}{2}$; $U_f = \left(-\left(\frac{M}{2}\right)g\frac{L}{8}\right) - \left(\frac{M}{2}g\frac{L}{4}\right)$



$\therefore\ W_{ext} = U_f - U_i = \frac{5}{16}MgL$

Example 50.



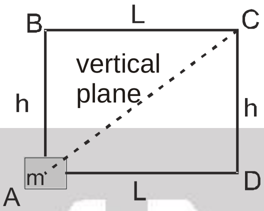
Find out the work done by spring force from A to B and from B to C. x = 0 is position of natural length.

Solution : $(W_{spring})_{A\to B} = U_i - U_f = \frac{1}{2}K\,(0.2)^2 - \frac{1}{2}K(0.1)^2$

$\therefore\ (W_{spring})_{A\to B} = \frac{3}{2}\,J$

Similarly $(W_{spring})_{BC} = \frac{1}{2}\,J$

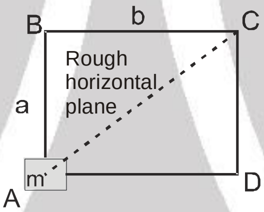
Example 51. **(a)** The mass m is moved from A to C along three different paths



(i) ABC (ii) ADC (iii) AC

Find out work done by gravity in the three cases.

(b) The block is moved from A to C along three different paths. Applied force is horizontal. Find work done by friction force in path



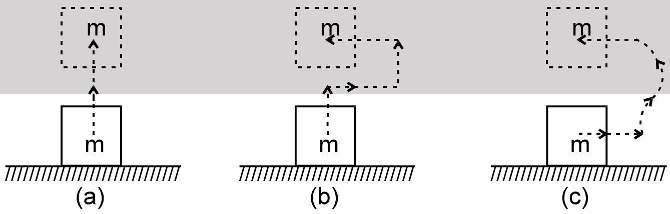
(i) ABC (ii) ADC (iii) AC

Solution : **(a)** (i) –mgh (ii) –mgh (iii) –mgh

(b) (i) $W_{ABC} = -\mu mg\,(a+b)$ (ii) $W_{ADC} = -\mu mg\,(a+b)$ (iii) $W_{AC} = -\mu mg\,(\sqrt{a^2+b^2})$

CONSERVATIVE FORCES

A force is said to be conservative if work done by or against the force in moving a body depends only on the initial and final positions of the body and not on the nature of path followed between the initial and final positions.



Consider a body of mass m being raised to a height h vertically upwards as show in above figure. The work done is *mgh*. Suppose we take the body along the path as in (b). The work done during horizontal motion is zero. Adding up the works done in the two vertical parts of the paths, we get the result *mgh* once again. Any arbitrary path like the one shown in (c) can be broken into elementary horizontal and vertical portions. Work done along the horizontal parts is zero. The work done along the vertical parts add up to *mgh.* Thus we conclude that the work done in raising a body against gravity is independent of the path taken. It only depends upon the initial and final positions of the body. We conclude from this discussion that the force of gravity is a conservative force.

Examples of Conservative forces.

(i) Gravitational force, not only due to the Earth but in its general form as given by the universal law of gravitation, is a conservative force.

(ii) Elastic force in a stretched or compressed spring is a conservative force.

(iii) Electrostatic force between two electric charges is a conservative force.

(iv) Magnetic force between two magnetic poles is a conservative forces.

In fact, all fundamental forces of nature are conservative in nature.

Forces acting along the line joining the centres of two bodies are called central forces. Gravitational force and Electrostatic forces are two important examples of central forces. Central forces are conservative forces.

PROPERTIES OF CONSERVATIVE FORCES

(i) Work done by or against a conservative force depends only on the initial and final positions of the body.

(ii) Work done by or against a conservative force does no depend upon the nature of the path between initial and final positions of the body.

If the work done a by a force in moving a body from an initial location to a final location is independent of the path taken between the two points, then the force is conservative.

(iii) Work done by or against a conservative force in a round trip is zero.

If a body moves under the action of a force that does no total work during any round trip, then the force is conservative; otherwise it is non-conservative.

The concept of potential energy exists only in the case of conservative forces.

(iv) The work done by a conservative force is completely recoverable.

Complete recoverability is an important aspect of the work of a conservative force.

NON-CONSERVATIVE FORCES

A force is said to be non-conservative if work done by or against the force in moving a body depends upon the path between the initial and final positions.

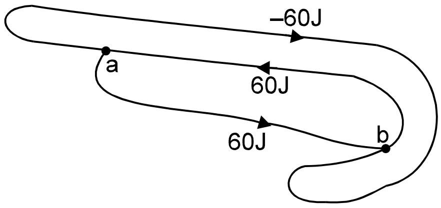
The frictional forces are non-conservative forces. This is because the work done against friction depends on the length of the path along which a body is moved. It does not depend only on the initial and final positions. Note that the work done by frictional force in a round trip is not zero.

The velocity-dependent forces such as air resistance, viscous force etc., are non conservative forces.

S.No.	Conservative forces	Non-Conservative forces
1	Work done does not depend upon path	Work done depends on path.
2	Work done in round trip is zero.	Work done in a round trip is not zero.
3	Central in nature.	Forces are velocity-dependent and retarding in nature.
4	When only a conservative force acts within a systrem, the kinetic enrgy and potential energy can change. However their sum, the mechanical energy of the system, does not change.	Work done against a non-conservative force may be dissipated as heat energy.
5	Work done is completely recoverable.	Work done in not completely recoverable.

Solved Example

Example 52. The figure shows three paths connecting points a and b. A single force F does the indicated work on a particle moving along each path in the indicated direction. On the basis of this information, is force F conservative?



Answer : No

Explanation : For a conservative force, the work done in a round trip should be zero.

Example 53. Find the work done by a force $\vec{F} = x\hat{i} + y\hat{j}$ acting on a particle to displace it from point A(0, 0) to B(2, 3).

Solution : $dW = \vec{F}.ds = (x\hat{i} + y\hat{j}).(dx\hat{i} + dy\hat{j})$

$$W = \int_0^2 xdx + \int_0^3 ydy = \left[\frac{x^2}{2}\right]_0^2 + \left[\frac{y^2}{2}\right]_0^3 = \frac{13}{2} \text{ units}$$

True or False

Example 54. In case of a non conservative force work done along two different paths will always be different.

Answer : False

Example 55. In case of non conservative force work done along two different paths may be different.

Answer : True

Example 56. In case of non conservative force work done along all possible paths cannot be same.

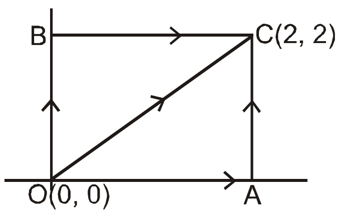
Answer : True

Example 57. Find work done by a force $\vec{F} = x\hat{i} + xy\hat{j}$ acting on a particle to displace it from point O(0, 0) to C(2, 2).

Solution : $\int dW = \int_0^2 xdx + \int_0^2 xydy$

can be found — cannot be found until x is known in terms of y i.e. until equation of path is known.

Example 58. Find the work done by $\vec{F}$ from O to C for above example if paths are given as below.



Solution : OAC $\Rightarrow$ OA + AC

for OA y = 0 $\therefore$ dy = 0

$\therefore \int dW_{OA} = \int_0^2 xdx + 0$ $\therefore$ $W_{OA} = 2$ J

for AC x = 2 dx = 0

$\int dW_{AC} = 0 + 2\int_0^2 ydy$ $\therefore$ $W_{AC} = 4J$

$W_{OAC} = W_{OA} + W_{AC} = 2 + 4 = 6J$

(ii) OBC $\Rightarrow$ OB + BC

for OB x = 0 dx = 0 $\therefore$ $W_{OB} = O$

for BC y = 2 dy = 0

$\therefore \int dW = \int xdx$ $\therefore$ $W = \left[\frac{x^2}{2}\right]_0^2 = 2$ J

$\therefore$ $W_{OAC} \neq W_{OBC}$

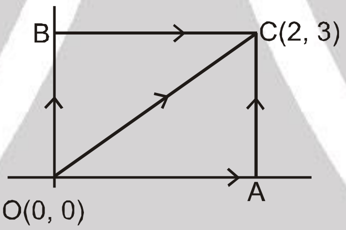
Hence the force is non-conservative.

(iii) For W_{OC} dW = xdy + xydx

for OC x = y dx = dy

$dW = \int_0^2 xdx + \int_0^2 y^2dy$ $W = \frac{14}{3}$ unit

Example 59. Find out work done by the force $\vec{F} = y\hat{i} + x\hat{j}$ to displace the particle from point O to C along the given paths. Decide whether the force is conservative or non–conservative.



Solution : (i) OAC OA + AC

for OA y = 0 dy = 0

$\therefore$ dW = 0 $W_{OA} = 0$

for AC x = 2 dx = 0

$\int dW = 2\int_0^3 dy$ $\Rightarrow$ W = 6 J $\Rightarrow$ $W_{OAC} = 6$ units

(ii) OBC $\Rightarrow$ OB + BC

for OB x = 0 dx = 0 $\therefore$ dW = 0

for BC y = 3 dy = 0

$\int dW = \int_0^2 3dx$ $\Rightarrow$ W = 6 units $\Rightarrow$ $W_{OBC} = 6$ units

(iii) OC

for OC $y = \frac{3}{2}x$ $dy = \frac{3}{2}dx$

$\therefore \int dW = \int_0^2 \frac{3}{2}xdx + \int_0^2 \frac{3}{2}xdx \Rightarrow \int dW = 3\int_0^2 x\,dx \Rightarrow$ $W_{OC} = 6$ units

Above force seems conservative but cannot be confirmed yet unless we can integrate it without the knowledge of path. Again we had dw = xdy + ydx & xdy + ydx can be written as dxy

$\therefore \int dW = \int dxy \Rightarrow W = \int_{0,0}^{2,3} dxy = [xy]_{0,0}^{2,3} = 6J$

Hence knowledge of path was not required to integrate the above so F is conservative.

POTENTIAL ENERGY AND CONSERVATIVE FORCE :

$F_S = -\partial U/\partial s$,

i.e., the projection of the field force, the vector **F**, at a given point in the direction of the displacement dr equals the derivative of the potential energy U with respect to a given direction, taken with the opposite sign. The designation of a partial derivative $\partial/\partial s$ emphasizes the fact of deriving with respect to a definite direction.

So, having reversed the sign of the partial derivatives of the function U with respect to x, y, z, we obtain the projection F_x, F_y and F_z of the vector **F** on the unit vectors **i**, **j** and **k**. Hence, one can readily find the vector itself : $F = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$, or

When conservative force does positive work then PE decreases

$$dU = -dw$$

$$dU = -F.ds$$

$$dU = -(F_x\hat{i} + F_y\hat{j} + F_z\hat{k}) \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$dU = -F_x dx - F_y dy - F_z dz$$

if y & z are constants then dy = 0 dz = 0

$$dU = -F_x dx$$

$\therefore \quad F_x = -\dfrac{dU}{dx}$ if y & z are constant

$$\equiv F_x = \frac{-\partial U}{\partial x}$$

Similarly $F_y = \dfrac{-\partial u}{\partial y}$; $F_z = \dfrac{-\partial u}{\partial z}$

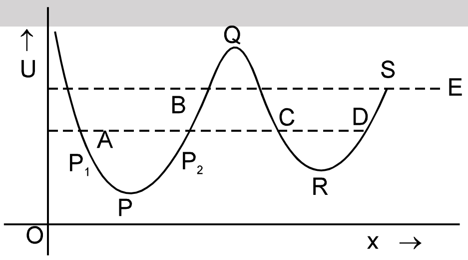
$$F = -\left(\frac{\partial U}{\partial x}\mathbf{i} + \frac{\partial U}{\partial y}\mathbf{j} + \frac{\partial U}{\partial z}\mathbf{k}\right).$$

The quantity in parentheses is referred to as the scalar gradient of the function U and is denoted by grad U or ∇U. We shall use the second, more convenient, designation where ∇ ("nabla") signifies the symbolic vector or operator

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

Potential Energy curve :

- A graph plotted between the PE a particle and its displacement from the centre of force field is called PE curve.



- Using graph, we can predict the rate of motion of a particle at various positions.
- Force on the particle is $F_{(x)} = -\dfrac{dU}{dx}$

Case-I : On increasing x, if U increases, force is in (–) ve x direction i.e. attraction force.

Case-II : On increasing x, if U decreases, force is in (+) ve x-direction i.e. repulsion force.

Solved Examples

Example 60. The potential energy of spring is given by $U = \frac{1}{2}kx^2$, where x is extension spring. Find the force associated with this potential energy.

Solution : $F_x = \frac{-\partial u}{\partial x} = -kx$ $\quad F_y = 0$ $\quad F_z = 0$.

Example 61. The potential energy of a particle in a space is given by $U = x^2 + y^2$. Find the force associated with this potential energy.

Solution : $F_x = \frac{-\partial u}{\partial x} = -[2x + 0] = -2x$

$F_y = \frac{-\partial u}{\partial y} = -(2y + 0) = -2y$; $\vec{F} = -2x\hat{i} - 2y\hat{j}$

Example 62. Find out the potential energy of given force $\vec{F} = -2x\hat{i} - 2y\hat{j}$.

Solution : $dU = -dW$

$\int dU = \int -(-2x\hat{i} - 2y\hat{j}) \bullet (dx\hat{i} + dy\hat{j})$

$\int dU = \int 2xdx + \int 2ydy$ $\quad \therefore \quad$ $\mathbf{U = x^2 + y^2 + C}$

Example 63. Find out the potential energy of the force $F = y\,\hat{i} + x\,\hat{j}$.

Solution : $dU = -dW$

$dU = -(y\hat{i} + x\hat{j}) \cdot (dx\hat{i} + dy\hat{j})$

$\int dU = \int -ydx + \int -xdy$

$\int dU = -\int dxy$ $\quad \Rightarrow \quad$ $U = -xy + c$

Example 64. Find out the force for which potential energy $U = -xy$.

Solution : $\vec{F} = -\left[\frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j}\right]$ $\quad \Rightarrow \quad$ $\vec{F} = -\left[\frac{\partial(-xy)}{\partial x}\hat{i} + \frac{\partial(-xy)}{\partial y}\hat{j}\right]$

$\vec{F} = y\hat{i} + x\hat{j}$ $\quad$ Hence verifying the previous example.

EQUILIBRIUM OF A PARTICLE

Different positions of a particle :

Position of equilibrium : If net force acting on a body is zero, it is said to be in equilibrium. For equilibrium $\frac{dU}{dx} = 0$. Points P, Q & R are the states of equilibrium positions.

Types of equilibrium :

- **Stable equilibrium :** When a particle is displaced slightly from a position and a force acting on it brings it back to the initial position, it is said to be in stable equilibrium position.

  Necessary conditions : $-\frac{dU}{dx} = 0$, and $\frac{d^2U}{dx^2} = +ve$

- **Unstable Equilibrium :** When a particle is displaced slightly from a position and force acting on it tries to displace the particle further away from the equilibrium position, it is said to be in unstable equilibrium.

  Condition : $-\frac{dU}{dx} = 0$ potential energy is maximum i.e. $= \frac{d^2U}{dx^2} = -$ ve

- **Neutral equilibrium :** In the neutral equilibrium potential energy is constant. When a particle is displaced from its position it does not experience any force acting on it and continues to be in equilibrium in the displaced position. This is said to be neutral equilibrium.

  A particle is in equilibrium if the acceleration of the particle is zero. As acceleration is frame dependent quantity therefore equilibrium depends on motion of observer also.

Solved Examples

Example 65. The potential energy between two atoms in a molecule is given by, $U_{(x)} = \frac{a}{x^{12}} - \frac{b}{x^6}$, where a and b are positive constants and x is the distance between the atoms. The system is in stable equilibrium when -

(A) $x = 0$ (B) $x = \frac{a}{2b}$ (C) $x = \left(\frac{2a}{b}\right)^{1/6}$ (D) $x = \left(\frac{11a}{5b}\right)$

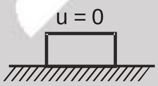
Answer: **(C)**

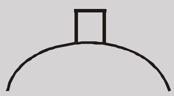
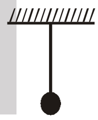
Solution : Given that, $U(x) = \frac{a}{x^{12}} - \frac{b}{x^6}$

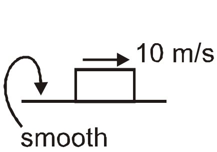
We, know $F = -\frac{du}{dx} = (-12)\, a\, x^{-13} - (-6b)\, x^{-7} = 0$

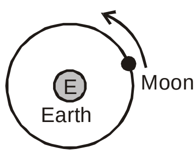
or $\frac{-6b}{x^7} = \frac{12a}{x^{13}}$ or $x^6 = 12a/6b = 2a/b$ or $x = \left(\frac{2a}{b}\right)^{1/6}$

$\frac{d^2U}{dx^2} = +\text{ve}$ at $x = \left(\frac{2a}{b}\right)^{1/6}$

Example 66. (A) 

(B) 
(C) 
(D) 

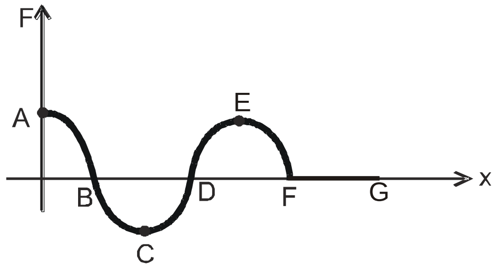
(E) 

(F) 

of the cases above which is not a case of equilibrium.

Solution : (F) as moon is always accelerated. It has centripetal acceleration or it is changing its velocity all the time.

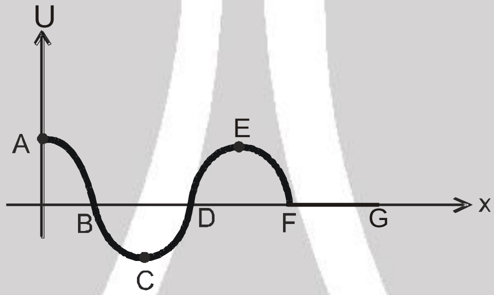
Example 67.



Find out positions of equilibrium and determine whether they are stable, unstable or neutral.

Solution : Equilibrium is at B, D, F as force is zero here.
For checking type of equilibrium displace slightly.
We have B as stable equilibrium
D as unstable equilibrium and F as neutral equilibrium

Example 68.



Identify the points of equilibrium and discuss their nature.

Solution : C, E, F are points of equilibrium because $F = \frac{-\partial U}{\partial x}$
When slope of U - x curve is zero then F is zero.
Check stability through slopes at near by points.
If we move right then slope should be positive for stable equilibrium and vice versa. In short it is like a hill and plateau.

MECHANICAL ENERGY :

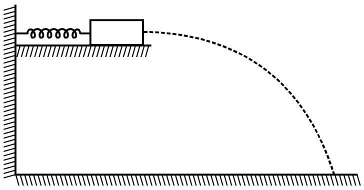
Definition: Mechanical energy 'E' of an object or a system is defined as the sum of kinetic energy 'K' and potential energy 'U', i.e., $E = K + U$

Important Points for M.E.:

1. It is a scalar quantity having dimensions $[ML^2T^{-2}]$ and SI units joule.
2. It depends on frame of reference.
3. A body can have mechanical energy without having either kinetic energy or potential energy. However, if both kinetic and potential energies are zero, mechanical energy will be zero. The converse may or may not be true, i.e., if E = 0 either both PE and KE are zero or PE may be negative and KE may be positive such that KE + PE = 0.
4. As mechanical energy E = K + U, i.e., E - U = K. Now as K is always positive, $E - U \geq 0$, i.e., for existence of a particle in the field, $E \geq U$.
5. As mechanical energy E = K + U and K is always positive, so, if 'U' is positive 'E' will be positive. However, if potential energy U is negative, 'E' will be positive if K > |U| and E will be negative if K < |U| i.e., mechanical energy of a body or system can be negative, and negative mechanical energy means that potential energy is negative and in magnitude it is more than kinetic energy. Such a state is called bound state, e.g., electron in an atom or a satellite moving around a planet are in bound state.

Solved Examples

Example 69. As shown in figure there is a spring block system. Block of mass 500 g is pressed against a horizontal spring fixed at one end to compress the spring through 5.0 cm. The spring constant is 500 N/m. When released, the block moves horizontally till it leaves the spring. Calculate the distance where it will hit the ground 4 m below the spring?



Solution : When block released, the block moves horizontally with speed V till it leaves the spring.

By energy conservation $\frac{1}{2}kx^2 = \frac{1}{2}mv^2$

$V^2 = \frac{kx^2}{m} \Rightarrow V = \sqrt{\frac{kx^2}{m}}$

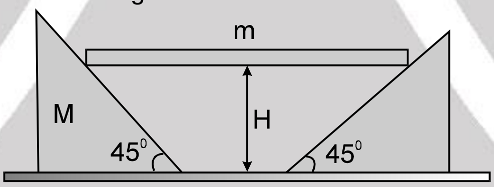
Time of flight $t = \sqrt{\frac{2H}{g}}$

So, horizontal distance travelled from the free end of the spring is V × t

$= \sqrt{\frac{kx^2}{m}} \times \sqrt{\frac{2H}{g}} = \sqrt{\frac{500 \times (0.05)^2}{0.5}} \times \sqrt{\frac{2 \times 4}{10}} = 2$ m

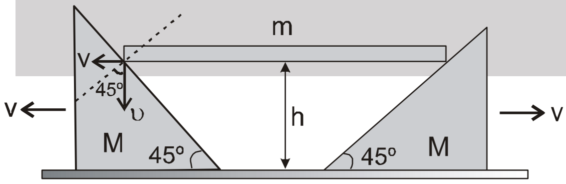
So, At a horizontal distance of 2 m from the free end of the spring.

Example 70. A rigid body of mass m is held at a height H on two smooth wedges of mass M each of which are themselves at rest on a horizontal frictionless floor. On releasing the body it moves down pushing aside the wedges. The velocity of recede of the wedges from each other when rigid body is at a height h from the ground is



(A) $\sqrt{\frac{2mg(H-h)}{m+2M}}$ (B) $\sqrt{\frac{2mg(H-h)}{2m+M}}$ (C) $\sqrt{\frac{8mg(H-h)}{m+2M}}$ (D) $\sqrt{\frac{8mg(H-h)}{2m+M}}$

Solution :



Let speed of the wedge and the rigid body be V and υ respectively.

Then applying wedge constraint we get

$V \cos 45^\circ = \upsilon \cos 45^\circ$

$\therefore \quad V = \upsilon$ ...(i)

Using energy conservation,

$mg(H-h) = 2\left(\frac{1}{2}MV^2\right) + \frac{1}{2}m\upsilon^2$ ...(ii)

From equation (i) and (ii)

$$V = \sqrt{\frac{2mg(H-h)}{m+2M}}$$

$\therefore$ The velocity of recede of wedges from each other = 2 × V = $\sqrt{\frac{8mg(H-h)}{m+2M}}$

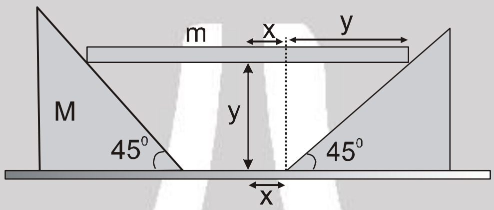
So, answer is **(C)**

Alter : Length of rod = ℓ

$x + y = \frac{\ell}{2}$ $\qquad\qquad \frac{dx}{dt} + \frac{dy}{dt} = 0$

velocity of block = velocity of rod

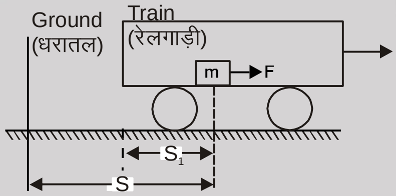
decrease in potential energy = increase in kinetic energy



$$mg\,(H-h) = \frac{1}{2}mV^2 + \frac{1}{2}MV^2 + \frac{1}{2}MV^2$$

$\therefore \quad V = \sqrt{\frac{2mg(H-h)}{2M+m}} \qquad \therefore \qquad 2V = \sqrt{\frac{8mg(H-h)}{2M+m}}$

Example 71. A block of mass m sits at rest on a frictionless table in a train that is moving with speed v_c (w.r.t. ground) along a straight horizontal track (fig.) A person in the train pushes on the block with a net horizontal force F for a time t in the direction of the car's motion.



(i) What is the final speed of the block according to a person in the train?

(ii) What is the final speed of the block according to a person standing on the ground outside the train?

(iii) How much did kinetic energy of the block change according to the person in the car?

(iv) How much did kinetic energy of the block change according to the person on the ground?

(v) In terms of F, m & t how far did the force displace the object according to the person in car?

(vi) According to the person on the ground?

(vii) How much work does each say the force did?

(viii) Compare the work done to the KE gain according to each person.

(ix) What can you conclude from this computation?

Answer : **(i)** $a_1 = F/m$, so $v_1 = a_1t = Ft/m$. **(ii)** Since velocities add, $v = v_c + v_1 = v_c + Ft/m$
(iii) $\Delta K_1 = m(v_1)^2/2 = F^2t^2/2m$ **(iv)** $\Delta K = m(v_c + v_1)^2/2 - mv_c^2/2$ **(v)** s_1 is $a_1t^2/2 = Ft^2/2m$

(vi) $s_1 + v_ct$ **(vii)** $W_g = F\left[V_c t + \frac{1}{2}\frac{Ft^2}{m}\right]$, $W_t = F\left[\frac{1}{2}\frac{F}{m}t^2\right]$

(viii) Compare W and W_1 with ΔK and ΔK_1, they are respectively equal.
(ix) The work - energy theorem holds for moving observers.

Solution : (i) w.r.t. person in the train $v_1 = at = \frac{Ft}{m}$

(ii) w.r.t. person on ground, $v = v_c + v_1 = v_c + \frac{Ft}{m}$

(iii) According to person in the train, $\Delta K_1 = \frac{1}{2}mv_1^2 = \frac{F^2t^2}{2m}$

(iv) According to person on ground, $\Delta K = \frac{1}{2}m.\left[v_c + \frac{Ft}{m}\right]^2 - \frac{1}{2}mv_c^2$

(v) $S_1 = \frac{1}{2}a_1t^2 = \frac{Ft^2}{2m}$.

(vi) According to person on ground, $S = v_ct + \frac{1}{2}\frac{F}{m}t^2 = \frac{Ft^2}{2m} + v_ct$.

(vii) According to person in the train work done by $F = Fs_1$

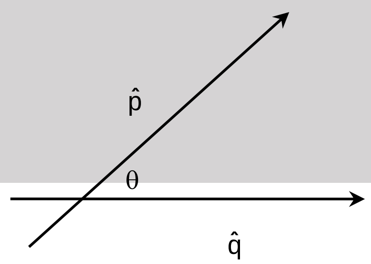
$$= \frac{F^2t^2}{2m}$$

According to person on ground,

Work done by F $= F.s = F\left[\frac{Ft^2}{2m} + v_ct\right]$.

(viii) Comparing $W_g = \Delta K_g$ and $W_c = \Delta K_c$.
(ix) Work–energy theorem holds in moving frame also.

Example 72. Motion of a particle in a plane is described by the non-orthogonal set of coordinates (p, q) with unit vectors $(\hat{p}, \hat{q})$ inclined at an angle θ as shown in the diagram. If the mass of the particle is m, its kinetic energy is given by $\left(\dot{x} = \frac{dx}{dt}\right)$ **[Olympiad (Stage-1) 2017]**

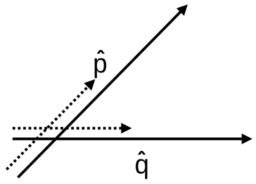


(A) $\frac{1}{2}m(\dot{p}^2 + \dot{q}^2 + \dot{p}\dot{q}\cos\theta)$ (B) $\frac{1}{2}m(\dot{p}^2 + \dot{q}^2 - \dot{p}\dot{q}(1 - \sin\theta))$

(C) $\frac{1}{2}m(\dot{p}^2 + \dot{q}^2 + 2\dot{p}\dot{q}\cos\theta)$ (D) $\frac{1}{2}m(\dot{p}^2 + \dot{q}^2 + \dot{p}\dot{q}\cot\theta)$

Answer : **(C)**

Solution :



$\vec{r} = (P\hat{P} + q\hat{q})$

$\vec{v} = \frac{d\vec{r}}{dt} = \frac{dp}{dt}\hat{p} + \frac{dq}{dt}\hat{q}$

$\vec{v} \cdot \vec{v} = \left(\frac{dp}{dt}\right)^2 + \left(\frac{dq}{dt}\right)^2 + 2pq\cos\theta$

K.E. = $\frac{1}{2}m\left(p^2 + q^2 + 2pq\cos\theta\right)$