# GRAVITATION

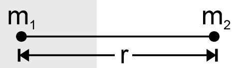
## 1. INTRODUCTION

The motion of celestial bodies such as the sun, the moon, the earth and the planets etc. has been a subject of fascination since time immemorial. Indian astronomers of the ancient times have done brilliant work in this field, the most notable among them being Aryabhata the first person to assert that earth rotates about its own axis.

A millennium later the Danish astronomer Tycho Brahe (1546-1601) conducted a detailed study of planetary motion which was interpreted by his pupil Johannes Kepler (1571-1630), ironically after the master himself had passed away. Kepler formulated his important findings in three laws of planetary motion. The basis of astronomy is gravitation.

## 2. UNIVERSAL LAW OF GRAVITATION : NEWTON'S LAW

**According to this law** "Each particle attracts every other particle. The force of attraction between them is directly proportional to the product of their masses and inversely proportional to square of the distance between them".




$F \propto \frac{m_1 m_2}{r^2}$ or $F = G\frac{m_1 m_2}{r^2}$

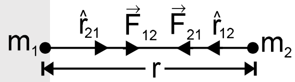
where $G = 6.67 \times 10^{-11}$ $Nm^2$ $kg^{-2}$ is the universal gravitational constant.

**Dimensional formula of G :**

$$F = \frac{Fr^2}{m_1 m_2} = \frac{[MLT^{-2}][L^2]}{[M^2]} = [M^{-1} L^3 T^{-2}]$$

**Newton's Law of gravitation in vector form :**

$$\vec{F}_{12} = -\frac{Gm_1 m_2}{r^2}\hat{r}_{12} \qquad \& \qquad \vec{F}_{21} = -\frac{Gm_1 m_2}{r^2}\hat{r}_{21}$$




Where $\vec{F}_{12}$ is the force on mass $m_1$ exerted by mass $m_2$ and vice-versa.

$\hat{r}_{12}$ : position vector of $m_1$ w.r.t. $m_2$ and vice–versa

Now $\hat{r}_{12} = -\hat{r}_{21}$, Thus $\vec{F}_{21} = \frac{Gm_1 m_2}{r^2}\hat{r}_{12}$. Comparing above, we get $\vec{F}_{12} = -\vec{F}_{21}$

**Important characteristics of gravitational force**

(i) Gravitational force between two bodies form an action and reaction pair i.e. the forces are equal in magnitude but opposite in direction.

(ii) Gravitational force is a central force i.e. it acts along the line joining the centers of the two interacting bodies.

(iii) Gravitational force between two bodies is independent of the nature of the medium, in which they lie.

(iv) Gravitational force between two bodies does not depend upon the presence of other bodies.

(v) Gravitational force is negligible in case of light bodies but becomes appreciable in case of massive bodies like stars and planets.

(vi) Gravitational force is long range-force i.e., gravitational force between two bodies is effective even if their separation is very large. For example, gravitational force between the sun and the earth is of the order of $10^{27}$ N although distance between them is $1.5 \times 10^8$ km

## Solved Example

**Example 1.** The centres of two identical spheres are at a distance 1.0 m apart. If the gravitational force between them is 1.0 N, then find the mass of each sphere. ($G = 6.67 \times 10^{-11}\ m^3\ kg^{-1}\ sec^{-2}$)

**Solution :** Gravitational force $F = \frac{Gm.m}{r^2}$

on substituting F = 1.0 N , r = 1.0 m and $G = 6.67 \times 10^{-11}\ m^3\ kg^{-1}\ sec^{-1}$

we get $m = 1.224 \times 10^5$ kg

**Example 2.** Three identical bodies of mass M are located at the vertices of an equilateral triangle with side L. At what speed must they move if they all revolve under the influence of one another's gravity in a circular orbit circumscribing the triangle while still preserving the equilateral triangle ?

**Solution :** Let A, B and C be the three masses and O the centre of the circumscribing circle. The radius of this circle is

$$R = \frac{L}{2}\sec 30° = \frac{L}{2} \times \frac{2}{\sqrt{3}} = \frac{L}{\sqrt{3}}.$$

Let v be the speed of each mass M along the circle. Let us consider the motion of the mass at A. The force of gravitational attraction on it due to the masses at B and C are

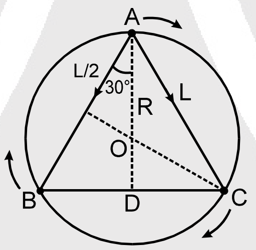
$\frac{GM^2}{L^2}$ along AB and $\frac{GM^2}{L^2}$ along AC

The resultant force is therefore

$$2\frac{GM^2}{L^2}\cos 30° = \frac{\sqrt{3}GM^2}{L^2} \text{ along AD.}$$

This, for preserving the triangle, must be equal to the necessary centripetal force.

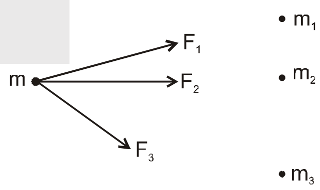
That is,




$$\frac{\sqrt{3}GM^2}{L^2} = \frac{Mv^2}{R} = \frac{\sqrt{3}Mv^2}{L} \qquad [\because R = L/\sqrt{3}\,] \quad \text{or} \quad v = \sqrt{\frac{GM}{L}}$$

## 2.1. Principle of superposition

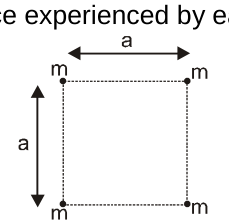
The force exerted by a particle on other particle remains unaffected by the presence of other nearby particles in space. Total force acting on a particle is the vector sum of all the forces acted upon by the individual masses when they are taken alone.



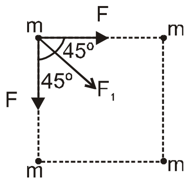

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + .......$$

**Solved Example**

**Example 1.** Four point masses each of mass 'm' are placed on the corner of square of side 'a'. Calculate magnitude of gravitational force experienced by each particle.



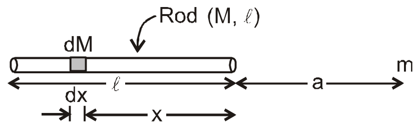
**Solution :**




$F_r$ = resultant force on each particle = 2F cos 45° + $F_1$

$$= \frac{2G.m^2}{a^2}\cdot\frac{1}{\sqrt{2}} + \frac{Gm^2}{(\sqrt{2}a)^2} = \frac{G.m^2}{2a^2}\,(2\sqrt{2}+1)$$

**Example 2.** Find gravitational force exerted by point mass 'm' on a uniform rod of mass 'M' and length '$\ell$'

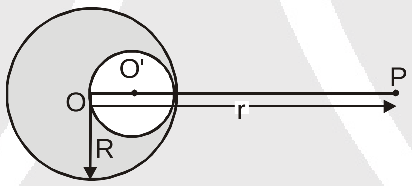



**Solution :** dF = force on element in horizontal direction = $\frac{G\cdot dM\cdot m}{(x+a)^2}$

where dM = $\frac{M}{\ell}$dx.

$$\therefore\quad F = \int dF = \int_0^{\ell}\frac{G.Mmdx}{\ell(x+a)^2} = \frac{G.Mm}{\ell}\int_0^{\ell}\frac{dx}{(x+a)^2} = \frac{G.Mm}{\ell}\left[-\frac{1}{(\ell+a)}+\frac{1}{a}\right] = \frac{GMm}{(\ell+a)a}$$

**Example 3.** A solid sphere of lead has mass M and radius R. A spherical portion is dug out from it (see figure) such that the boundary of hollowed part passes through the centre and also touches the boundary of the solid sphere. Deduce the gravitational force on a mass m placed at P, which is distant r from O along the line of centres.




**Solution :** Let O be the centre of the sphere and O' that of the hollow (figure). For an external point the sphere behaves as if its entire mass is concentrated at its centre. Therefore, the gravitational force on a mass `m` at P due to the original sphere (of mass M) is

$$F = G\frac{Mm}{r^2}\text{, along PO.}$$

The diameter of the smaller sphere (which would be cut off) is R, so that its radius OO' is R/2. The force on m at P due to this sphere of mass M' (say) would be

$$F' = G\frac{M'm}{(r-\frac{R}{2})^2} \text{ along PO}'. \qquad [\because \text{ distance PO}' = r - R/2]$$

As the radius of this sphere is half of that of the original sphere, we have
M′ = M/8.

$$\therefore\quad F' = G\frac{Mm}{8(r-\frac{R}{2})^2} \text{ along PO}'.$$

As both F and F′ point along the same direction, the force due to the remaining hollowed sphere is

$$F - F' = \frac{GMm}{r^2} - \frac{GMm}{8r^2(1-\frac{R}{2r})^2} = \frac{GMm}{r^2}\left\{1-\frac{1}{8(1-\frac{R}{2\,r})^2}\right\}.$$

# 3. GRAVITATIONAL FIELD

The space surrounding the body within which its gravitational force of attraction is experienced by other bodies is called gravitational field. Gravitational field is very similar to electric field in electrostatics where charge 'q' is replaced by mass 'm' and electric constant 'K' is replaced by gravitational constant 'G'. The intensity of gravitational field at a point is defined as the force experienced by a unit mass placed at that point.

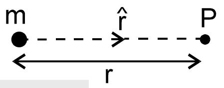
$$\vec{E} = \frac{\vec{F}}{m}$$

The unit of the intensity of gravitational field is N kg$^{-1}$.

**Intensity of gravitational field due to point mass :**

The force due to mass m on test mass $m_0$ placed at point P is given by :

$$F = \frac{GMm_0}{r^2}$$




Hence $E = \dfrac{F}{m_0} \Rightarrow E = \dfrac{GM}{r^2}$

In vector form $\vec{E} = -\dfrac{GM}{r^2}\hat{r}$

Dimensional formula of intensity of gravitational field = $\dfrac{F}{m} = \dfrac{[MLT^{-2}]}{[M]} = [M^0LT^{-2}]$

## Solved Example

**Example 1.** Find the distance of a point from the earth's centre where the resultant gravitational field due to the earth and the moon is zero. The mass of the earth is 6.0 × 10$^{24}$ kg and that of the moon is 7.4 × 10$^{22}$ kg. The distance between the earth and the moon is 4.0 × 10$^5$ km.

**Solution :** The point must be on the line joining the centres of the earth and the moon and in between them. If the distance of the point from the earth is x km, the distance from the moon is (4.0 × 10$^5$ – x) km. The magnitude of the gravitational field due to the earth is

$$E_1 = \frac{GM_e}{x^2} = \frac{G \times 6 \times 10^{24}\,kg}{x^2}$$

and magnitude of the gravitational field due to the moon is

$$E_2 = \frac{GM_m}{(4.0 \times 10^5 - x)^2} = \frac{G \times 7.4 \times 10^{22}\,kg}{(4.0 \times 10^5 - x)^2}$$

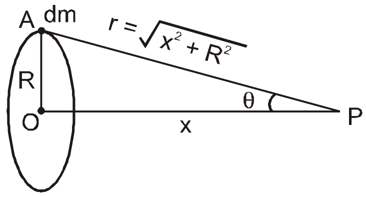
These fields are in opposite directions. For the resultant field to be zero $|E_1| = |E_2|$

or, $\dfrac{6 \times 10^{24}\,kg}{x^2} = \dfrac{7.4 \times 10^{22}\,kg}{(4.0 \times 10^5 - x)^2}$ or, $\dfrac{x}{4.0 \times 10^5 - x} = \sqrt{\dfrac{6 \times 10^{24}}{7.4 \times 10^{22}}} = 9$

or, x = 3.6 × 10$^5$ km.

**Example 2.** Calculate gravitational field intensity due to a uniform ring of mass M and radius R at a distance x on the axis from center of ring.

**Solution :** Consider any particle of mass dm. Gravitational field at point P due to dm




$dE = \dfrac{Gdm}{r^2}$ along PA

Component along PO is $dE \cos\theta = \dfrac{Gdm}{r^2}\cos\theta$

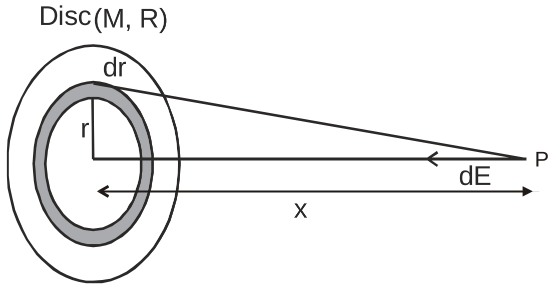
Net gravitational field at point P is

$$E = \int \frac{Gdm}{r^2}\cos\theta = \frac{G\cos\theta}{r^2}\int dm$$

$$= \frac{GMx}{\left(R^2 + x^2\right)^{3/2}}$$ towards the center of ring

**Example 3.** Calculate gravitational field intensity at a distance x on the axis from centre of a uniform disc of mass M and radius R.

**Solution :** Consider a elemental ring of radius r and thickness dr on surface of disc as shown in figure




Gravitational field due to elemental ring

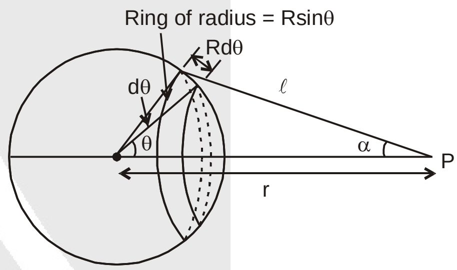
$$dE = \frac{GdMx}{(x^2+r^2)^{3/2}} \quad \text{Here } dM = \frac{M}{\pi R^2}\,.\,2\pi r dr = \frac{2M}{R^2} r dr$$

$$\therefore \quad dE = \frac{G.2Mxrdr}{R^2(x^2+r^2)^{3/2}}$$

$$\therefore \quad E = \int_0^R \left(\frac{2GMx}{R^2}\right)\frac{rdr}{(x^2+r^2)^{3/2}} \qquad \therefore \quad E = \frac{2GMx}{R^2}\left[\frac{1}{x} - \frac{1}{\sqrt{x^2+R^2}}\right]$$

**Example 4.** For a given uniform spherical shell of mass M and radius R, find gravitational field at a distance r from centre in following two cases (a) $r \geq R$ (b) $r < R$

**Solution :** Field at P due to an elemental ring




$$dE = \frac{GdM}{\ell^2}\,.\cos\alpha \qquad r \geq R$$

$$dM = \frac{M}{4\pi R^2} \times 2\pi R \sin\theta \, Rd\theta$$

$$dM = \frac{M}{2}\sin\theta \, d\theta$$

$$\therefore \quad dE = \frac{GM \sin\theta \cos\alpha \, d\theta}{2\ell^2}$$

Now $\ell^2 = R^2 + r^2 - 2rR\cos\theta$ .....(1)

$R^2 = \ell^2 + r^2 - 2\ell r \cos\alpha$ .....(2)

$$\therefore \quad \cos\alpha = \frac{\ell^2+r^2-R^2}{2\ell r}$$

$$\cos\theta = \frac{R^2+r^2-\ell^2}{2rR}$$

differentiating (1)

$\therefore \quad 2\ell \, d\ell = 2rR\sin\theta \, d\theta$

$$\therefore \quad dE = \frac{GM}{2\ell^2}\,.\,\frac{\ell d\ell}{Rr}\,.\,\frac{\ell^2+r^2-R^2}{(2\ell r)} \quad \Rightarrow \quad dE = \frac{GM}{4Rr^2}\left[1+\frac{r^2-R^2}{\ell^2}\right]d\ell$$

$$\therefore \quad E = \int dE = \frac{GM}{4Rr^2}\left[\int_{r-R}^{r+R} d\ell + (r^2-R^2)\int_{r-R}^{r+R}\frac{d\ell}{\ell^2}\right] \quad \Rightarrow \quad E = \frac{GM}{r^2}, \quad r \geq R$$

If point is inside the shell limit changes to [(R – r) to R + r]
E = 0 when r < R.

**Example 5.** Find the relation between the gravitational field on the surface of two planets A & B of masses $m_A$, $m_B$ & radius $R_A$ & $R_B$ respectively if
(i) They have equal mass
(ii) They have equal (uniform) density

**Solution :** Let $E_A$ & $E_B$ be the gravitational field intensities on the surface of planets A & B.

then, $E_A = \dfrac{Gm_A}{R_A^2} = \dfrac{G\frac{4}{3}\pi R_A^3 \rho_A}{R_A^2} = \dfrac{4G\pi}{3}\rho_A R_A$

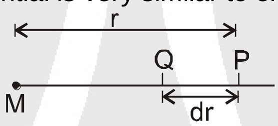
Similarly, $E_B = \dfrac{Gm_B}{R_B{}^2} = \dfrac{4G}{3}\pi\ \rho_B R_B$

(i) For $m_A = m_B$, $\dfrac{E_A}{E_B} = \dfrac{R_B^2}{R_A^2}$

(ii) For $\rho_A = \rho_B$, $\dfrac{E_A}{E_B} = \dfrac{R_A}{R_B}$

# 4. GRAVITATIONAL POTENTIAL

The gravitational potential at a point in the gravitational field of a body is defined as the amount of work done by an external agent in bringing a body of unit mass from infinity to that point, slowly (no change in kinetic energy). Gravitational potential is very similar to electric potential in electrostatics.




**Gravitational potential due to a point mass:**

Let the unit mass be displaced through a distance dr towards mass M,

then work done is given by $dW = F\,dr = \dfrac{GM}{r^2}dr$

Total work done in displacing the particle from infinity to point P is $W = \int dW = \int_\infty^r \dfrac{GM}{r^2}dr = \dfrac{-GM}{r}$.

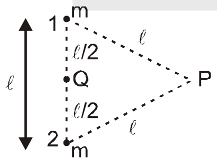
Thus gravitational potential, $\boxed{V = -\dfrac{GM}{r}}$.

The unit of gravitational potential is J kg$^{-1}$. Dimensional Formula of gravitational potential

$= \dfrac{\text{Work}}{\text{mass}} = \dfrac{[ML^2T^{-2}]}{[M]} = [M^0L^2T^{-2}]$.

## Solved Example

**Example 1.**




Find out potential at P and Q due to the two point mass system. Find out work done by external agent in bringing unit mass from P to Q. Also find work done by gravitational force.

**Solution :**

(i) $V_{P1}$ = potential at P due to mass 'm' at '1' = $-\dfrac{Gm}{\ell}$

$V_{P2} = -\dfrac{Gm}{\ell}$

$\therefore\quad V_P = V_{P1} + V_{P2} = -\dfrac{2Gm}{\ell}$

(ii) $V_{Q1} = -\frac{GM}{\ell/2} \Rightarrow V_{Q2} = -\frac{Gm}{\ell/2}$

$\therefore V_Q = V_{Q1} + V_{Q2} = -\frac{Gm}{\ell/2} - \frac{Gm}{\ell/2} = -\frac{4Gm}{\ell}$

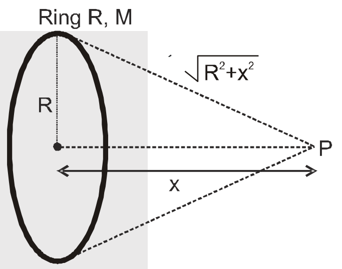
Force at point Q = 0

(iii) work done by external agent = $(V_Q - V_P) \times 1 = -\frac{2GM}{\ell}$

(iv) work done by gravitational force = $V_P - V_Q = \frac{2GM}{\ell}$

**Example 2.** Find potential at a point 'P' at a distance 'x' on the axis away from centre of a uniform ring of mass M and radius R.

**Solution :** Ring can be considered to be made of large number of point masses ($m_1$, $m_2$ ..........etc)




$V_P = -\frac{Gm_1}{\sqrt{R^2+x^2}} - \frac{Gm_2}{\sqrt{R^2+x^2}} - .......$

$= -\frac{G}{\sqrt{R^2+x^2}}(m_1 + m_2.....) = -\frac{GM}{\sqrt{R^2+x^2}}$,

where $M = m_1 + m_2 + m_3 + ............$

Potential at centre of ring = $-\frac{G.M}{R}$

## 5. RELATION BETWEEN GRAVITATIONAL FIELD AND POTENTIAL

The work done by an external agent to move unit mass from a point to another point in the direction of the field E, slowly through an infinitesimal distance dr = Force by external agent × distance moved = – Edr.

Thus $dV = -Edr$

$\Rightarrow E = -\frac{\partial V}{\partial r}$ ($\because$ V can also be a function of $\theta$)

Therefore, gravitational field at any point is equal to the negative gradient at that point.

### Solved Example

**Example 1.** The gravitational field in a region is given by $\vec{E} = 20(\hat{i} + \hat{j})$ N/kg. Find the gravitational potential difference (in J/kg) between the points A(5m, 4m) and the origin (0, 0).

(A) – 180 (B) 180 (C) – 90 (D) zero

**Answer :** **(A)**

**Solution :** $V_A - V_0 = -\int_{(0,0)}^{(5,4)} 20(\hat{i} + \hat{j}).(dx\hat{i} + dy\hat{j}) = -20\,[(5-0) + (4-0)] = -180$ J/kg

**Example 2.** In the above problem, find the work to be done in slowly shifting a particle of mass 1 kg from origin (0, 0) to a point (5, 4): (In J)

(A) – 180 (B) 180 (C) – 90 (D) zero

**Answer :** (A)

**Solution :** $W = m(V_f - V_i) = m(V_A - V_0) = 1\ (-180) = -180$ J

**Example 3.** $v = 2x^2 + 3y^2 + zx$, find gravitational field at a point (x, y, z).

**Solution :** $E_x = \dfrac{-\partial v}{\partial x} = -4x - z$

$E_y = -6y$

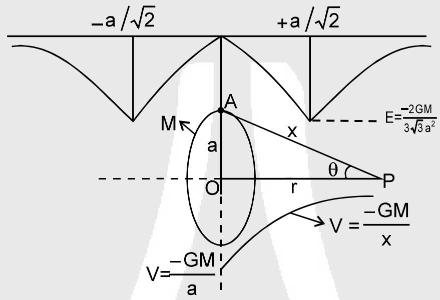
$E_z = -x$

$\therefore$ Field $= \vec{E} = -[(4x + z)\,\hat{i} + 6y\,\hat{j} + x\,\hat{k}]$.

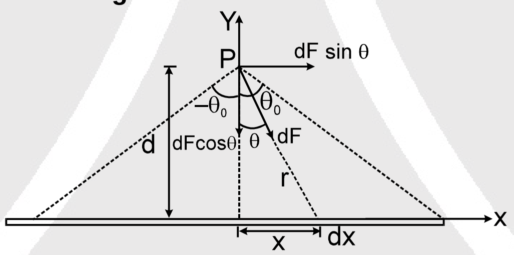
# 6. GRAVITATIONAL POTENTIAL & FIELD FOR DIFFERENT OBJECTS

**I. Ring.** $V = \dfrac{-GM}{x}$ or $\dfrac{-GM}{(a^2 + r^2)^{1/2}}$ & $E = \dfrac{-GMr}{(a^2 + r^2)^{3/2}}\hat{r}$ or $E = -\dfrac{GM\cos\theta}{x^2}$

Gravitational field is maximum at a distance, $r = \pm\, a/\sqrt{2}$ and it is $-2GM/3\sqrt{3}\,a^2$




**II. A linear mass of finite length on its axis :**




**(a) Potential** :

$\Rightarrow\ V = -\dfrac{GM}{L}\,\ell n\,(\sec\theta_0 + \tan\theta_0) = -\dfrac{GM}{L}\,\ell n\left\{\dfrac{L + \sqrt{L^2 + d^2}}{d}\right\}$

**(b) Field intensity :**

$\Rightarrow\ E = -\dfrac{GM}{Ld}\sin\theta_0 = -\dfrac{GM}{d\sqrt{L^2 + d^2}}$

**III. An infinite uniform linear mass distribution of linear mass density $\lambda$, Here $\theta_0 = \dfrac{\pi}{2}$.**

And noting that $\lambda = \dfrac{M}{2L}$ in case of a finite rod

we get, for field intensity $E = \dfrac{2G\lambda}{d}$

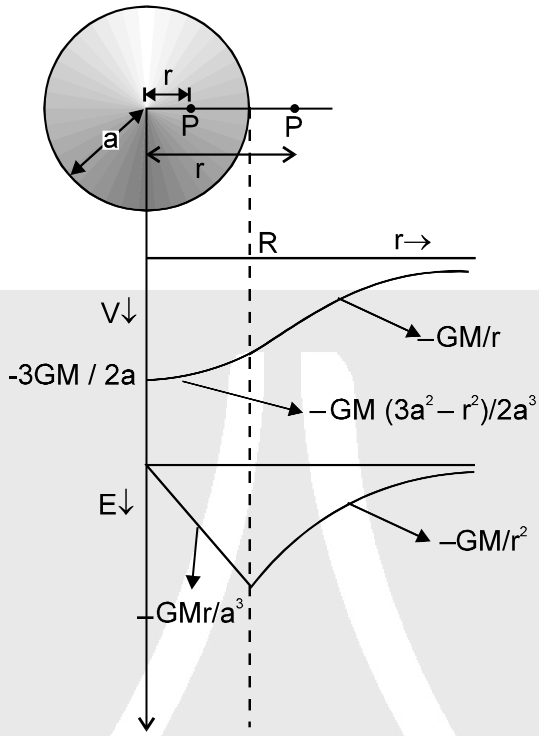
Potential for a mass-distribution extending to infinity is not defined. However even for such mass distributions potential-difference is defined. Here potential difference between points $P_1$ and $P_2$ respectively at distances $d_1$ and $d_2$ from the infinite rod, $v_{12} = 2G\lambda\,\ell n\,\dfrac{d_2}{d_1}$

**IV. Uniform Solid Sphere**

**(a) Point P inside the shell.** $r \le a$, then

$$V = -\frac{GM}{2a^3}(3a^2 - r^2) \;\; \& \;\; E = -\frac{GMr}{a^3}\text{, and at the centre } V = -\frac{3GM}{2a} \text{ and } E = 0$$

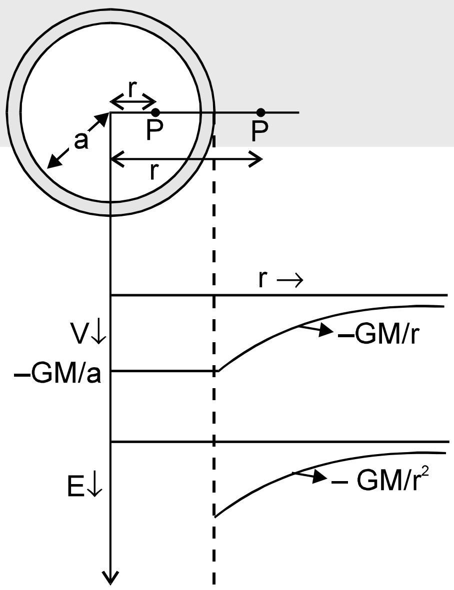
**(b) Point P outside the shell.** $r \ge a$, then $V = -\frac{GM}{r}$ & $E = -\frac{GM}{r^2}$




**V. Uniform Thin Spherical Shell**

**(a) Point P Inside the shell.** $r \le a$, then $V = \frac{-GM}{a}$ & $E = 0$

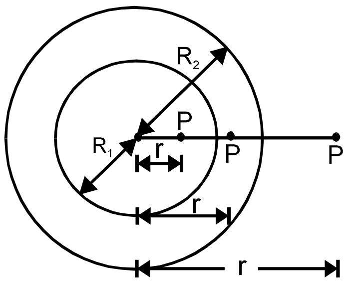
**(b) Point P outside shell.** $r \ge a$, then $V = \frac{-GM}{r}$ & $E = -\frac{GM}{r^2}$



**VI. Uniform Thick Spherical Shell**

**(a) Point outside the shell** $V = -G\frac{M}{r}$ ; $E = -G\frac{M}{r^2}$

**(b) Point inside the Shell (Inside the cavity)** $V = -\frac{3}{2}GM\left(\frac{R_2 + R_1}{R_2^2 + R_1R_2 + R_1^2}\right)$

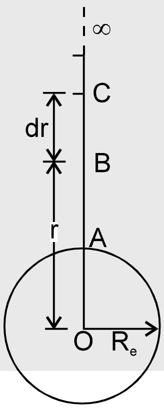



E = 0

**(c) Point between the two surface** $V = -\frac{GM}{2r}\left(\frac{3rR_2^2 - r^3 - 2R_1^3}{R_2^3 - R_1^3}\right)$; $E = -\frac{GM}{r^2}\frac{r^3 - R_1^3}{R_2^3 - R_1^3}$

# 7. GRAVITATIONAL POTENTIAL ENERGY

Gravitational potential energy of two mass system is equal to the work done by an external agent in assembling them, while their initial separation was infinity. Consider a body of mass m placed at a distance r from another body of mass M. The gravitational force of attraction between them is given by, $F = \frac{GMm}{r^2}$.




Now, Let the body of mass m is displaced from point C to B through a distance 'dr' towards the mass M, then work done by internal conservative force (gravitational) is given by,

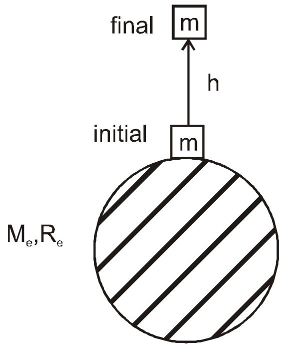
$$dW = F\,dr = \frac{GMm}{r^2}\,dr$$

$$\Rightarrow \int dW = \int_{\infty}^{r} \frac{GMm}{r^2}\,dr$$

$\therefore$ Gravitational potential energy, $\boxed{U = -\frac{GMm}{r}}$

### Increase in gravitational potential energy:

Suppose a block of mass m on the surface of the earth. We want to slowly lift this block by 'h' height.




Work required in this process = increase in P.E. = $U_f - U_i = m(V_f - V_i)$

$$W_{ext} = \Delta U = (m)\left[-\left(\frac{GM_e}{R_e+h}\right)-\left(-\frac{GM_e}{R_e}\right)\right]$$

$$W_{ext} = \Delta U = GM_e m\left(\frac{1}{R_e}-\frac{1}{R_e+h}\right)$$

$$= \frac{GM_e m}{R_e}\left(1-\left(1+\frac{h}{R_e}\right)^{-1}\right)$$

(as h << $R_e$, we can apply Bionomial theorem)

$$W_{ext} = \Delta U = \frac{GM_e m}{R_e}\left(1-\left(1-\frac{h}{R_e}\right)\right) = (m)\left(\frac{GM_e}{R_e{}^2}\right)h$$

$$W_{ext} = \Delta U = mgh$$

* This formula is valid only when h << $R_e$

## Solved Example

**Example 1.** A body of mass m is placed on the surface of earth. Find work required to lift this body by a height

(i) $h = \frac{R_e}{1000}$

(ii) $h = R_e$

**Solution :**

(i) $h = \frac{R_e}{1000}$, as h << $R_e$, so

we can apply

$W_{ext} = U\uparrow = mgh$

$$W_{ext} = (m)\left(\frac{GM_e}{R_e{}^2}\right)\left(\frac{R_e}{1000}\right) = \frac{GM_e m}{1000R_e}$$

(ii) $h = R_e$, in this case h is not very less than $R_e$, so we cannot apply $\Delta U = mgh$

so we cannot apply $\Delta U = mgh$

$W_{ext} = U\uparrow = U_f - U_i = m(V_f - V_i)$

$$W_{ext} = m\left[\left(-\frac{GM_e}{R_e+R_e}\right)-\left(-\frac{GM_e}{R_e}\right)\right]$$

$$W_{ext} = -\frac{GM_e m}{2R_e}$$

**Example 2.** Calculate the velocity with which a body must be thrown vertically upward from the surface of the earth so that it may reach a height of 10 R, where R is the radius of the earth and is equal to $6.4 \times 10^6$ m. (Earth's mass = $6 \times 10^{24}$ kg, Gravitational constant $G = 6.7 \times 10^{-11}$ N-m$^2$/kg$^2$)

**Solution :** The gravitational potential energy of a body of mass m on earth's surface is

$$U(R) = -\frac{GMm}{R}$$

where M is the mass of the earth (supposed to be concentrated at its centre) and R is the radius of the earth (distance of the particle from the centre of the earth). The gravitational energy of the same body at a height 10 R from earth's surface, i.e. at a distance 11R from earth's centre is

$$U(11\,R) = -\frac{GMm}{11R}$$

$\therefore$ Change in potential energy $U(11\,R) - U(R) = -\frac{GMm}{11R} - \left(-\frac{GMm}{R}\right) = \frac{10}{11}\frac{GMm}{R}$

This difference must come from the initial kinetic energy given to the body in sending it to that height. Now, suppose the body is thrown up with a vertical speed v, so that its initial kinetic energy is $\frac{1}{2}mv^2$. Then $\frac{1}{2}mv^2 = \frac{10}{11}\frac{GMm}{R}$ or $v = \sqrt{\frac{20}{11}\frac{GM}{R}}$

Putting the given values : $v = \sqrt{\left(\frac{20 \times (6.7 \times 10^{-11} N - m^2/kg^2) \times (6 \times 10^{24} kg)}{11\ (6.4 \times 10^6 m)}\right)} = 1.07 \times 10^4$ m/s.

**Example 3.** Two particles of masses $m_1$ and $m_2$, initially at rest at infinite distance from each other, move under the action of mutual gravitational pull. Show that at any instant their relative velocity of approach is $\sqrt{2G(m_1 + m_2)/R}$ , where R is their separation at that instant.

**Solution :**

**Method 1 : (Force method)**

The gravitational force of attraction on $m_1$ due to $m_2$ at a separation r is $F_1 = \frac{Gm_1m_2}{r^2}$

Therefore, the acceleration of $m_1$ is $a_1 = \frac{F_1}{m_1} = \frac{Gm_2}{r^2}$

Similarly, the acceleration of $m_2$ due to $m_1$ is $a_2 = -\frac{Gm_1}{r^2}$

the negative sign being put as $a_2$ is directed opposite to $a_1$. The relative acceleration of approach is

$$a = a_1 - a_2 = \frac{G(m_1 + m_2)}{r^2} \quad .....(1)$$

If v is the relative velocity, then $a = \frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt}$.

But $-\frac{dr}{dt} = v$ (negative sign shows that r decreases with increasing t).

$$\therefore \quad a = -\frac{dv}{dr}\ v. \quad .....(2)$$

From (1) and (2), we have $v\,dv = -\frac{G(m_1 + m_2)}{r^2}dr$

Integrating, we get $\frac{v^2}{2} = \frac{G(m_1 + m_2)}{r} + C$

At $r = \infty$, v = 0 (given), and so C = 0.

$$\therefore \quad v^2 = \frac{2G(m_1 + m_2)}{r}$$

Let $v = v_R$ when r = R. Then $v_R = \sqrt{\left(\frac{2G(m_1 + m_2)}{R}\right)}$

**Method 2 : (Momentum and energy conservation method) :**

Since, the particles are moving under their mutual interaction only, thus $F_{external} = 0$, conserving momentum at the time of release and at the instant when separation becomes R, we get

$m_1v_1 - m_2v_2 = 0 \quad \Rightarrow \quad v_2 = \dfrac{m_1v_1}{m_2}$ …(i)

Conserving total mechanical energy : $0 = \dfrac{1}{2}m_1v_1^2 + \dfrac{1}{2}m_2v_2^2 - \dfrac{Gm_1m_2}{R}$

using equation (i) : $\dfrac{1}{2}m_1v_1^2\left(1+\dfrac{m_1}{m_2}\right) = \dfrac{Gm_1m_2}{R} \Rightarrow v_1 = m_2\sqrt{\dfrac{2G}{(m_1+m_2)R}}$

Similarly, $v_2 = m_1\sqrt{\dfrac{2G}{(m_1+m_2)R}}$

Therefore, the relative velocity is $v_R = v_1 + v_2 = \sqrt{\left(\dfrac{2G(m_1+m_2)}{R}\right)}$

# 8. GRAVITATIONAL SELF-ENERGY

The gravitational self-energy of a body (or a system of particles) is defined as the work done by an external agent in assembling the body (or system of particles) from infinitesimal elements (or particles) that are initially an infinite distance apart.

## 8.1. Gravitational self-energy of a system of n particles

Potential energy of n particles due to their mutual gravitational attraction is equal to the sum of the potential energy of all pairs of particle, i.e.,

$U_S = -G\sum_{j=1}^{n}\sum_{\substack{j=1\\ j>i}}^{n}\dfrac{m_im_j}{r_{ij}}$ , where $r_{ij}$ is the distance between the $i^{th}$ and $j^{th}$ particles

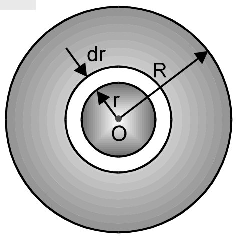
This expression can be written as $U_s = -\dfrac{1}{2}G\sum_{i=1}^{i=n}\sum_{\substack{j=1\\ j\neq i}}^{j=n}\dfrac{m_im_j}{r_{ij}}$ (1/2 factor arises due to the fact that all the terms have been considered twice)

If we consider a system of 'n' particles, each of same mass 'm' and separated from each other by the same average distance 'r', then self energy

or $\quad U_s = -\dfrac{1}{2}G\sum_{i=1}^{n}\sum_{\substack{j=1\\ j\neq i}}^{n}\left(\dfrac{m^2}{r}\right)_{i\,j}$

Thus on the right hand side 'i' comes 'n' times while 'j' comes (n – 1) times. Thus $U_s = -\dfrac{1}{2}Gn(n-1)\dfrac{m^2}{r}$

## 8.2. Gravitational Self energy of a Uniform Sphere (star)




$U_{sphere} = -G\dfrac{\left(\frac{4}{3}\pi r^3\rho\right)\left(4\pi r^2 dr\rho\right)}{r}$ where $\rho = \dfrac{M}{\left(\frac{4}{3}\right)\pi R^3}$

$= -\dfrac{1}{3}G(4\pi\rho)^2 r^4\,dr,$

$U_{star} = -\dfrac{1}{3}G\,(4\pi\rho)^2\int_0^R r^4 dr = -\dfrac{1}{3}G\,(4\pi\rho)^2\left[\dfrac{r^5}{5}\right]_0^R = -\dfrac{3}{5}G\left(\dfrac{4\pi}{3}R^3\rho\right)^2\dfrac{1}{R}.$

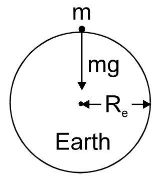
$\therefore \quad U_{star} = -\dfrac{3}{5}\dfrac{GM^2}{R}$

# 9. ACCELERATION DUE TO GRAVITY :

It is the acceleration, a freely falling body near the earth's surface acquires due to the earth's gravitational pull. The property by virtue of which a body experiences or exerts a gravitational pull on another body is called **gravitational mass $m_G$**, and the property by virtue of which a body opposes any change in its state of rest or uniform motion is called its **inertial mass $m_I$** thus if $\vec{E}$ is the gravitational field intensity due to the earth at a point P, and $\vec{g}$ is acceleration due to gravity at the same point, then $m_I\vec{g} = m_G\vec{E}$.

Now the value of inertial & gravitational mass happens to be exactly same to a great degree of accuracy for all bodies. Hence, $\vec{g} = \vec{E}$

The gravitational field intensity on the surface of earth is therefore numerically equal to the acceleration due to gravity (g) there. Thus we get,




$$g = \frac{GM_e}{R_e^{\,2}}$$

where, $M_e$ = Mass of earth
$R_e$ = Radius of earth

**Note :**

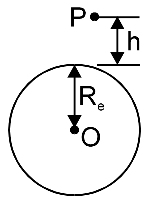
- Here the distribution of mass in the earth is taken to be spherically symmetric so that its entire mass can be assumed to be concentrated at its center for the purpose of calculation of g. Also we have considered uniform average density throughout the whole volume of earth.

# 10. VARIATION OF ACCELERATION DUE TO GRAVITY

## 10.1. Effect of Altitude

Acceleration due to gravity on the surface of the earth is given by, $g = \frac{GM_e}{R_e^2}$

Now, consider the body at a height 'h' above the surface of the earth, then the acceleration due to gravity at height 'h' given by

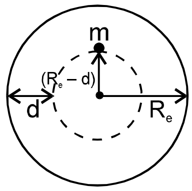



$$g_h = \frac{GM_e}{(R_e+h)^2} = g\left(1+\frac{h}{R_e}\right)^{-2} \simeq g\left(1-\frac{2h}{R_e}\right) \text{ when } h << R.$$

The decrease in the value of 'g' with height h = $g - g_h = \frac{2gh}{R_e}$. Then percentage decrease in the value of 'g' $= \frac{g-g_h}{g}\times 100 = \frac{2h}{R_e}\times 100\%$

## 10.2. Effect of depth

The gravitational pull on the surface is equal to its weight i.e. $mg = \frac{GM_em}{R_e^2}$




$$\therefore\quad mg = \frac{G\times\frac{4}{3}\pi R_e^3\rho m}{R_e^2} \quad\text{or}\quad g = \frac{4}{3}\pi G R_e \rho \qquad .........(1)$$

When the body is taken to a depth d, the mass of the sphere of radius $(R_e - d)$ will only be effective for the gravitational pull and the outward shell will have no resultant effect on the mass. If the acceleration due to gravity on the surface of the solid sphere is $g_d$, then

$$g_d = 4/3\,\pi G\,(R_e - d)\,\rho \qquad .......(2)$$

By dividing equation (2) by equation (1)

$$\Rightarrow\quad g_d = g\left(1-\frac{d}{R_e}\right)$$

**Important Points**

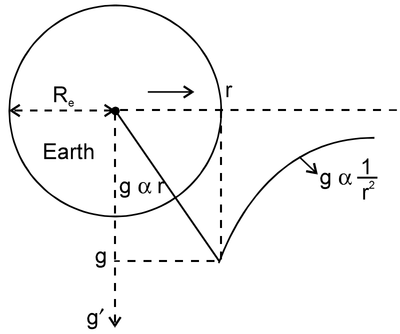
**(i)** At the center of the earth, d = $R_e$, so $g_{centre} = g\left(1 - \frac{R_e}{R_e}\right) = 0$.

Thus weight (mg) of the body at the centre of the earth is zero.

**(ii)** Percentage decrease in the value of 'g' with the depth

$$= \left(\frac{g - g_d}{g}\right) \times 100\ \%$$

$$= \frac{d}{R_e} \times 100\ \%$$




**Solved Example**

**Example 1.** The value of acceleration due to gravity at Earth's surface is 9.8 $ms^{-2}$. The altitude above its surface at which the acceleration due to gravity decreases to 4.9 $ms^{-2}$, is close to : (Radius of earth = $6.4 \times 10^6$ m)

(A) $2.6 \times 10^6$ m (B*) $6.4 \times 10^6$ m (C) $9.0 \times 10^6$ m (D) $1.6 \times 10^6$ m

**Solution :** $\frac{g}{2} = \frac{GM}{(R+h)^2} = \frac{gR^2}{(R+h)^2}$

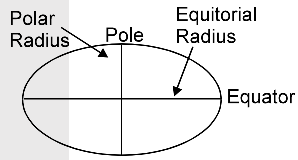
$R + h = \sqrt{2}\,R$

$R = 0.41\,R = 0.41 \times 6.4 \times 10^6\,m = 2.6 \times 10^6\,m$

## 10.3. Effect of the surface of Earth

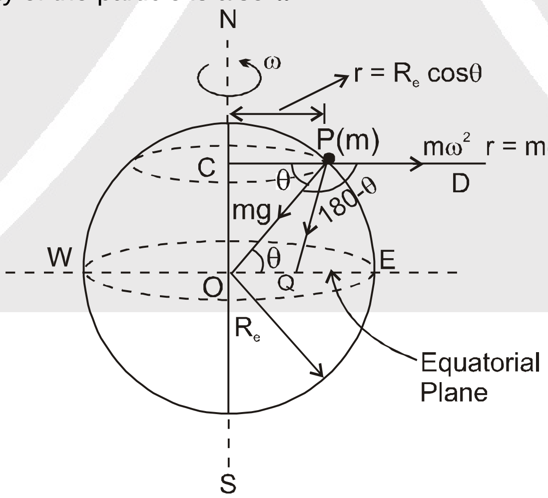
The equatorial radius is about 21 km longer than its polar radius.

We know, $g = \frac{GM_e}{R_e^2}$ Hence $g_{pole} > g_{equator}$. The weight of the body increase as the body taken from the equator to the pole.




## 10.4. Effect of rotation of the Earth

The earth rotates around its axis with angular velocity ω. Consider a particle of mass m at latitude θ. The angular velocity of the particle is also ω.




According to parallelogram law of vector addition, the resultant force acting on mass m along PQ is

$F = [(mg)^2 + (m\omega^2 R_e \cos\theta)^2 + \{2mg \times m\omega^2 R_e \cos\theta\} \cos(180 - \theta)]^{1/2}$

$= [(mg)^2 + (m\omega^2 R_e \cos\theta)^2 - (2m^2 g\omega^2 R_e \cos\theta) \cos\theta]^{1/2}$

$$= mg\left[1 + \left(\frac{R_e\omega^2}{g}\right)^2 \cos^2\theta - 2\frac{R_e\omega^2}{g}\cos^2\theta\right]^{1/2}$$

The value of $\omega = 7.3 \times 10^{-5}$ rad/s, thus we ignore the term containing $\omega^4$ and after binomial expansion and neglecting higher powers, we get

$$F \simeq mg\left[1-\frac{2R_e\omega^2\cos^2\theta}{g}\right]^{\frac{1}{2}} = mg\left[1-\frac{R_e\omega^2\cos^2\theta}{g}\right] \Rightarrow g' = \frac{F}{m} = g - \omega^2 R_e \cos^2\theta$$

At pole $\theta = 90° \Rightarrow g_{pole} = g$ ; At equator $\theta = 0° \Rightarrow g_{equator} = g\left[1-\frac{R_e\omega^2}{g}\right]$.

Hence $g_{pole} > g_{equator}$

If the body is taken from pole to the equator, then $g' = g\left(1-\frac{R_e\omega^2}{g}\right)$.

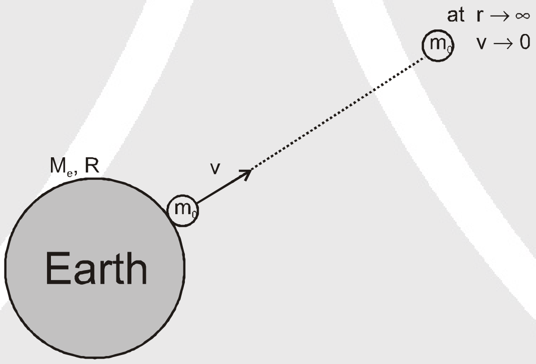
Hence percentage change in weight = $\dfrac{mg - mg\left(1-\frac{R_e\omega^2}{g}\right)}{mg} \times 100\% = \dfrac{mR_e\omega^2}{mg} \times 100\% = \dfrac{R_e\omega^2}{g} \times 100\%$

## 11. ESCAPE SPEED

The minimum speed required to send a body out of the gravity field of a planet (send it to $r \to \infty$)

### 11.1. Escape speed at earth's surface :

Suppose a particle of mass $m_0$ is on earth's surface. We project it with a velocity $v_e$ from the earth's surface, so that it just reaches $r \to \infty$ (at $r \to \infty$, its velocity become zero). Applying energy conservation between initial position (when the particle was at earth's surface) and final position (when the particle just reaches $r \to \infty$)




$K_i + U_i = K_f + U_f$

$$\frac{1}{2}m_0 v_e^2 + m_0\left(-\frac{GM_e}{R}\right) = 0 + m_0\left(-\frac{GM_e}{(r\to\infty)}\right) \Rightarrow v_e = \sqrt{\frac{2GM_e}{R}}$$

Escape speed from earth surface is $v_e = \sqrt{\frac{2GM_e}{R}}$

If we put the values of G, Me, R the we get $V_e$ = 11.2 km/s.

### 11.2. Escape speed depends on :

(i) Mass ($M_e$) and size (R) of the planet
(ii) Position from where the particle is projected.

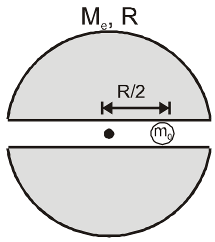
### 11.3. Escape speed does not depend on :

(i) Mass of the body which is projected ($m_0$)
(ii) Angle of projection.

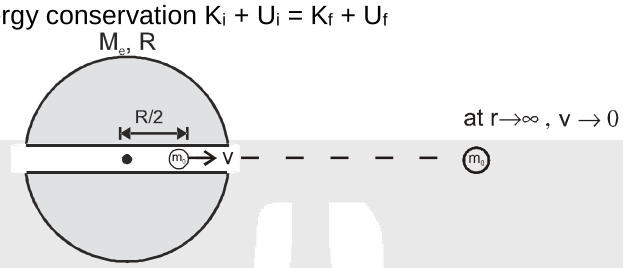
If a body is thrown from Earth's surface with escape speed, it goes out of earth's gravitational field and never returns to the earth's surface. But it starts revolving around the sun.

**Solved Example**

**Example 1.** A very thin groove is made in the earth along its diameter, and a particle of mass $m_0$ is placed at R/2 distance from the centre. Find the escape speed of the particle from that place.




**Solution :** Suppose we project the particle with speed v, so that it just reaches at ($r \to \infty$).
Applying energy conservation $K_i + U_i = K_f + U_f$




$$\frac{1}{2} m_0 v^2 + m_0 \left\{-\frac{GM_e}{2R^3}\left(3R^2 - \left(\frac{R}{2}\right)^2\right)\right\} = 0 + 0$$

$v = \sqrt{\frac{11GM_e}{4R}} = V_e$ at that position.

**Example 2.** Find radius of such planet on which the man escapes through jumping. The capacity of jumping of person on earth is 1.5 m. Density of planet is same as that of earth.

**Solution :** For a planet : $\frac{1}{2}mv^2 - \frac{GM_P m}{R_p} = 0 \quad \Rightarrow \frac{1}{2} mv^2 = \frac{GM_P m}{R_p}$

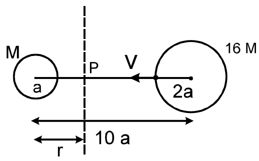
On earth $\rightarrow \frac{1}{2}mv^2 = m\left(\frac{GM_E}{R_E^2}\right)h$

$\therefore \quad \frac{GM_P m}{R_p} = \frac{GM_E . m}{R_E^2} . h \quad \Rightarrow \quad \frac{M_p}{R_p} = \frac{M_E h}{R_E^2}$

$\therefore$ Density ($\rho$) is same $\quad \Rightarrow \quad \frac{(4/3)\pi R_p^3 \rho}{R_p} = \frac{(4/3)\pi R_E^3 h\rho}{R_E^2} \quad \Rightarrow R_P = \sqrt{R_E h}$

**Example 3.** Distance between centres of two stars is 10 a. The masses of these stars are M and 16 M and their radii are a & 2a respectively. A body is fired straight from the surface of the larger star towards the smaller star. What should be its minimum initial speed to reach the surface of the smaller star?

**Solution :** Let P be the point on the line joining the centres of the two planets such that the net field at this point is zero




Then, $\frac{GM}{r^2} - \frac{G(16M)}{(10a - r)^2} = 0 \quad \Rightarrow \quad (10a - r)^2 = 16\,r^2$

$\Rightarrow \quad 10a - r = 4r \quad \Rightarrow \quad r = 2a$

Potential at point P, $V_p = -\frac{GM}{r} - \frac{G(16M)}{(10a - r)} = \frac{-GM}{2a} - \frac{2GM}{a} = \frac{-5GM}{2a}$.

Now, if the particle projected from the larger planet has enough energy to cross this point P, it will reach the smaller planet.

For this, the K.E. imparted to the body must be just enough to raise its total mechanical energy to a value which is equal to P.E. at point P.

i.e. $\frac{1}{2}mv^2 - \frac{G(16M)m}{2a} - \frac{GMm}{8a} = mV_p$

or, $\frac{v^2}{2} - \frac{8GM}{a} - \frac{GM}{8a} = -\frac{5GM}{2a}$

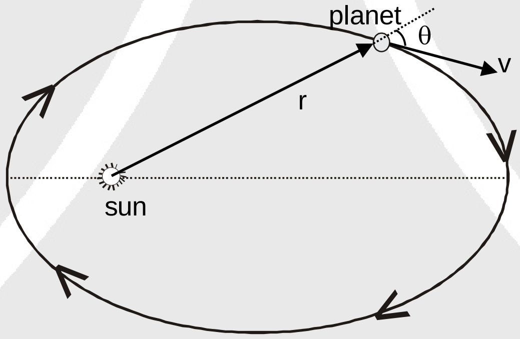
or, $v^2 = \frac{45GM}{4a}$ or, $v_{min} = \frac{3}{2}\sqrt{\frac{5GM}{a}}$

## 12. KEPLER’S LAW OF PLANETARY MOTION

Suppose a planet is revolving around the sun, or a satellite is revolving around the earth, then the planetary motion can be studied with help of Kepler’s three laws. We may treat the sun and planets as point masses as their separations are very large. The mas of sun being very large compared to any other object in the solar system, its motion is essentially unaffected by the gravity of the planets.

### 12.1. Kepler’s Law of orbit

Each planet moves around the sun in elliptical path with the sun at one of its focii. (In fact circular path is a subset of elliptical path)




### 12.2. Law of areal velocity :

To understand this law, lets understand the angular momentum conservation for the planet.

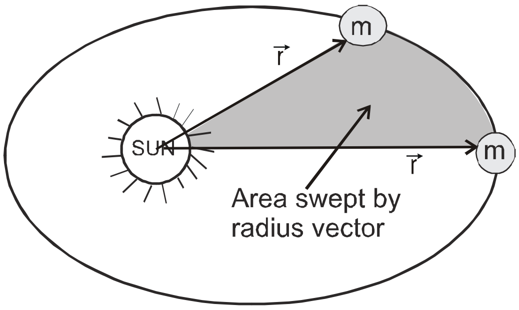
If a planet moves in an elliptical orbit, the gravitation force acting on it always passes through the centre of the sun. So torque of this gravitation force about the centre of the sun will be zero. Hence we can say that angular momentum of the planet about the centre of the sun will remain conserved (constant)

$\tau$ about the sun = 0

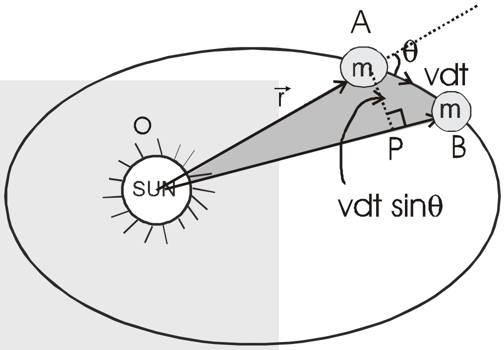
$\Rightarrow \frac{dL}{dt} = 0 \quad \Rightarrow \quad L_{planet,sun} = \text{constant} \quad \Rightarrow \quad mvr\sin\theta = \text{constant}$

Now we can easily study the Kepler's law of aerial velocity.

If a planet moves around the sun, the radius vector ($\vec{r}$) also rotates and sweeps area as shown in figure. Now lets find rate of area swept by the radius vector ($\vec{r}$).




Suppose a planet is revolving around the sun and at any instant its velocity is v, and angle between radius vector ($\vec{r}$) and velocity ($\vec{v}$). In dt time, it moves by a distance vdt, during this dt time, area swept by the radius vector will be OAB which can be assumed to be a triangle




dA = 1/2 (Base) (Perpendicular height)
dA = 1/2 (r) (vdtsinθ)

so rate of area swept $\frac{dA}{dt} = \frac{1}{2}vr\sin\theta$

we can write $\frac{dA}{dt} = \frac{1}{2}\frac{mvr\sin\theta}{m}$

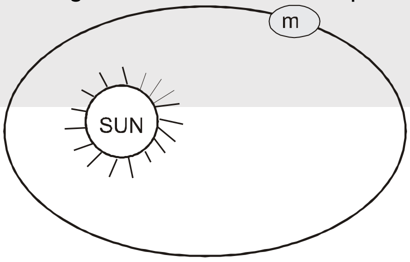
where mvr sinθ = angular momentum of the planet about the sun, which remains conserved (constant)

$\Rightarrow \frac{dA}{dt} = \frac{L_{planet/sun}}{2m}$ = constant

so **Rate of area swept by the radius vector is constant**

**Solved Example**

**Example 1.** Suppose a planet is revolving around the sun in an elliptical path given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Find time period of revolution. Angular momentum of the planet about the sun is L.




**Solution :** Rate of area swept $\frac{dA}{dt} = \frac{L}{2m}$ = constant

$\Rightarrow dA = \frac{L}{2m}dt$ ; $\int_{A=0}^{A=\pi ab} dA = \int_{t=0}^{t=T} \frac{L}{2m}dt$

$\Rightarrow \pi ab = \frac{L}{2m}T \Rightarrow T = \frac{2\pi mab}{L}$

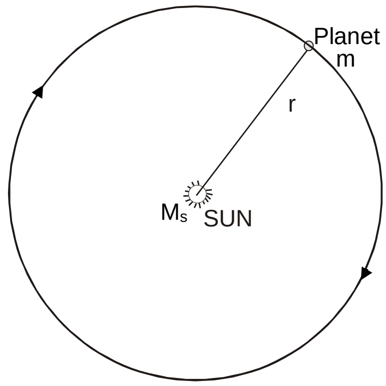
## 12.3. Kepler's law of time period :

Suppose a planet is revolving around the sun in circular orbit then $\frac{mv^2}{r} = \frac{GM_s m}{r^2}$

$$v = \sqrt{\frac{GM_s}{r}}$$

Time period of revolution is $T = \frac{2\pi r}{v} = 2\pi r\sqrt{\frac{r}{GM_s}}$

$$T^2 = \left(\frac{4\pi^2}{GM_s}\right)r^3 \Rightarrow T^2 \alpha\, r^3$$

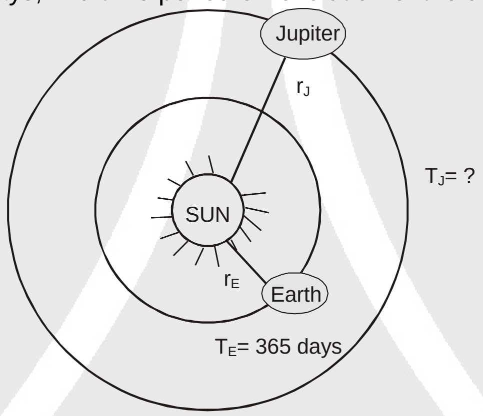



For all the planet of a sun, $T^2 \propto r^3$

* If planets are moving in elliptical orbit, then $T^2 \propto a^3$ where a = semi major axis of the elliptical path.

**Solved Example**

**Example 1.** The Earth and Jupiter are two planets of the sun. Consider the average orbital radius of the earth to be $1.5 \times 10^8$ km and that of Jupiter to be $6 \times 10^8$ km. If the time period of revolution of earth is T = 365 days, find time period of revolution of the Jupiter.



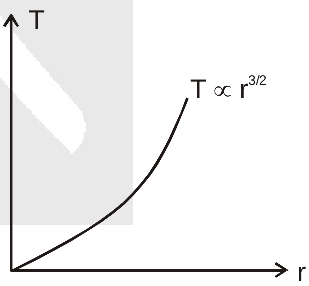

**Solution :** For both the planets

$T^2 \propto r^3$

$$\left(\frac{T_{Jupiter}}{T_{Earth}}\right)^2 = \left(\frac{r_{Jupiter}}{r_{Earth}}\right)^3 \Rightarrow \left(\frac{T_{Jupiter}}{365\text{days}}\right)^2 = \left(\frac{6\times10^8}{1.5\times10^8}\right)^3$$

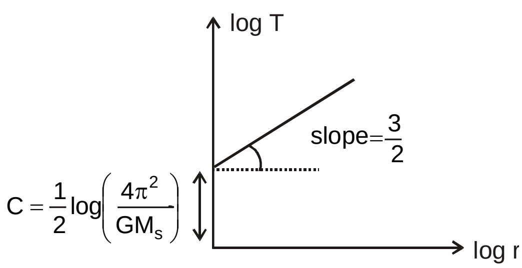
$T_{Jupiter} = 8 \times 365$ days

Graph of T vs r :

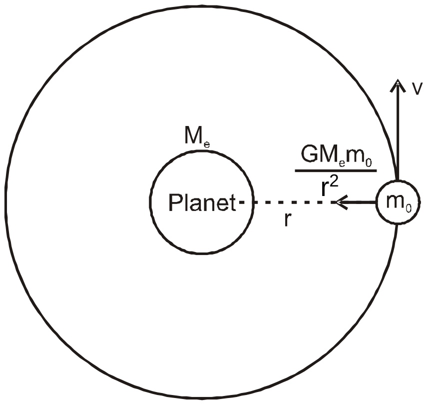



Graph of log T v/s log r :

$$T^2 = \left(\frac{4\pi^2}{GM_s}\right)r^3 \Rightarrow 2\log T = \log\left(\frac{4\pi^2}{GM_s}\right) + 3\log r \Rightarrow \log T = \frac{1}{2}\log\left(\frac{4\pi^2}{GM_s}\right) + \frac{3}{2}\log r$$



# 13. CIRCULAR MOTION OF A SATELLITE AROUND A PLANET




Suppose a satellite (e.g. natural satellite moon) of mass $m_0$ is at a distance r from a planet (e.g. earth). If the satellite does not revolve, then due to the gravitational attraction, it will fall onto surface of the planet.

To avoid the collision, the satellite revolve around the planet, for circular motion of satellite.

$\Rightarrow \quad \frac{GM_e m_0}{r^2} = \frac{m_0 v^2}{r}$ ....(1)

$\Rightarrow \quad v = \sqrt{\frac{GM_e}{r}}$ this velocity is called orbital velocity ($v_0$)

$v_0 = \sqrt{\frac{GM_e}{r}}$

## 13.1. Total energy of the satellite moving in circular orbit :

(i) KE = $\frac{1}{2} m_0 v^2$ and from equation (1)

$\frac{m_0 v^2}{r} = \frac{GM_e m_0}{r^2} \quad \Rightarrow \quad m_0 v^2 = \frac{GM_e m_0}{r} \quad \Rightarrow \quad KE = \frac{1}{2} m_0 v^2 = \frac{GM_e m_0}{2r}$

(ii) Potential energy $U = -\frac{GM_e m_0}{r}$

Total energy = KE + PE = $\left(\frac{GM_e m_0}{2r}\right) + \left(\frac{-GM_e m_0}{r}\right)$

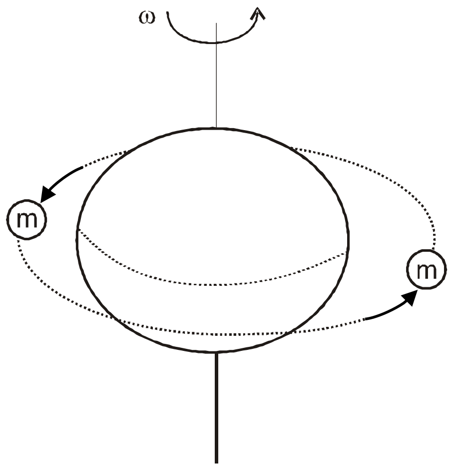
$TE = -\frac{GM_e m_0}{2r}$

Total energy is –ve. It shows that the satellite is still bounded with the planet.

# 14. GEO - STATIONARY SATELLITE :

We know that the earth rotates about its axis with angular velocity $\omega_{earth}$ and time period $T_{earth}$ = 24 hours.

Suppose a satellite is set in an orbit which is in the plane of the equator, whose $\omega$ is equal to $\omega_{earth}$, (or its T is equal to $T_{earth}$ = 24 hours) and direction is also same as that of earth. Then, when seen from earth, it will appear to be stationery. This type of satellite is called geo- stationary satellite. For a geo-stationary satellite,




$\omega_{satellite} = \omega_{earth}$

$\Rightarrow$ $T_{satellite} = T_{earth}$ = 24 hr.

So time period of a geo-stationery satellite must be 24 hours. To achieve T = 24 hour, the orbital radius geo-stationary satellite :

$$T^2 = \left(\frac{4\pi^2}{GM_e}\right) r^3$$

Putting the values, we get orbital radius of geo stationary satellite r = 6.6 $R_e$ (here Re = radius of the earth)

Height from the surface h = 5.6 $R_e \approx$ 36000 km

## Solved Example

**Example 1.** A satellite is launched into a circular orbit 1600 km above the surface of the earth. Find the period of revolution if the radius of the earth is R = 6400 km and the acceleration due to gravity on earth's surface is 9.8 m/s$^2$. At what height from the ground should it be launched so that it may appear stationary over a point on the earth's equator?

**Solution :** The orbiting period of a satellite at a height h from earth's surface is T = $\frac{2\pi r^{3/2}}{\sqrt{gR^2}}$ where r = R + h

then, T = $\frac{2\pi(R+h)}{R}\sqrt{\left(\frac{R+h}{g}\right)}$

Here, R = 6400 km, h = 1600 km = R/4.

Then T = $\frac{2\pi\left(R+\frac{R}{4}\right)}{R}\sqrt{\left(\frac{R+\frac{R}{4}}{g}\right)} = 2\pi(5/4)^{3/2}\sqrt{\frac{R}{g}}$

Putting the given values : T = 2 × 3.14 × $\sqrt{\left(\frac{6.4\times10^6 m}{9.8\ \ m/s^2}\right)}$ $(1.25)^{3/2}$ = 7096 sec = 1.97 hours

Now, a satellite will appear stationary in the sky over a point on the earth's equator if its period of revolution round the earth is equal to the period of revolution of the earth round its own axis which is 24 hours. Let us find the height h of such a satellite above the earth's surface in terms of the earth's radius. Let it be nR. Then

T = $\frac{2\pi(R+nR)}{R}\sqrt{\left(\frac{R+nR}{g}\right)} = 2\pi\sqrt{\left(\frac{R}{g}\right)}$ $(1 + n)^{3/2}$ = 2 × 3.14 $\sqrt{\left(\frac{6.4\times10^6 meter/sec}{9.8 meter/sec^2}\right)}$ $(1 + n)^{3/2}$

= (5075 sec) $(1 + n)^{3/2}$ = (1.41hours) $(1 + n)^{3/2}$

For T = 24 hours, we have

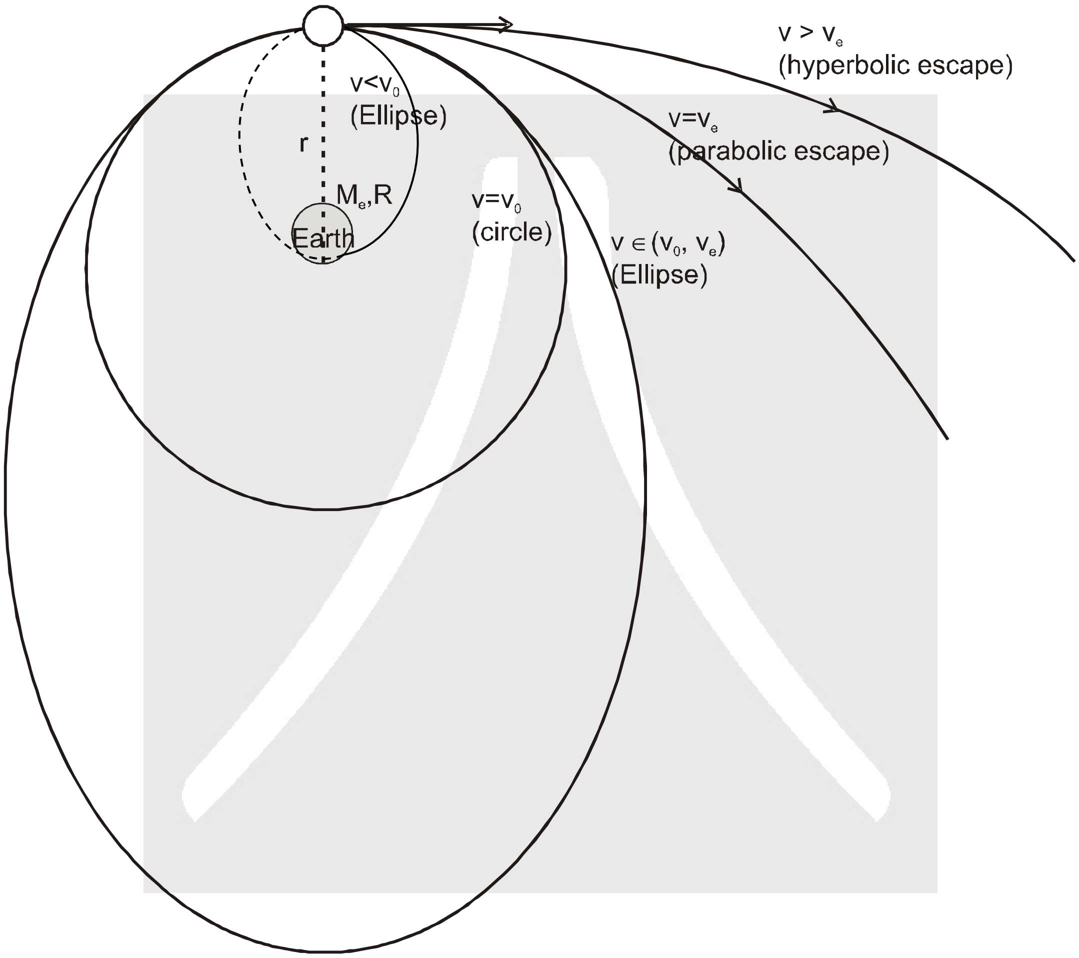
(24 hours) = (1.41 hours) $(1 + n)^{3/2}$

or $(1 + n)^{3/2} = \dfrac{24}{1.41} = 17$

or $1 + n = (17)^{2/3} = 6.62$ or $n = 5.62$

The height of the geo-stationary satellite above the earth's surface is nR = 5.62 × 6400 km = $3.595 \times 10^4$ km.

## 15. PATH OF A SATELLITE ACCORDING TO DIFFERENT SPEED OF PROJECTION




Suppose a satellite is at a distance r from the centre of the earth. If we give different velocities (v) to the satellite, its path will be different

(i) If $v < v_0$ $\left(\text{or} \quad v < \sqrt{\dfrac{GM_e}{r}}\right)$ then the satellite will move in an elliptical path and strike the earth's surface. But if size of earth were small or a thin elliptical chute is made in earth then the satellite would complete the elliptical orbit with the centre of the earth at its farther focus. Total energy of satellite is < 0.

(ii) If $v = v_0 \left( \text{or} \quad v = \sqrt{\frac{GM_e}{r}} \right)$, then the satellite will revolve in a circular orbit. Total energy of satellite is < 0.

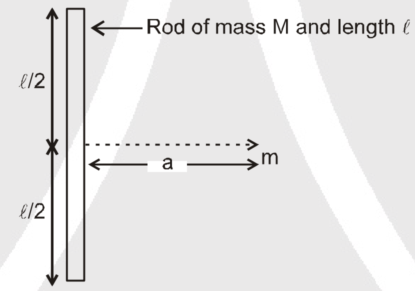
(iii) If $v_e > v > v_0 \left( \text{or} \quad \sqrt{\frac{2GM_e}{r}} > v > \sqrt{\frac{GM_e}{r}} \right)$, then the satellite will revolve in a elliptical orbital, and the centre of the earth will be at its nearer focus. Total energy of satellite is < 0.

(iv) If $v = v_e \left( \text{or} \quad v = \sqrt{\frac{2GM_e}{r}} \right)$, then the satellite will just escape to infinity along a parabolic path. Total energy of satellite is = 0.

(v) If $v > v_e \left( \text{or} \quad v > \sqrt{\frac{2GM_e}{r}} \right)$, then the satellite will again escape to infinity but along a hyperbolic path. Total energy of satellite is > 0.

## SOLVED MISCELLANEOUS PROBLEMS

**Problem 1.** Calculate the force exerted by point mass m on rod of uniformly distributed mass M and length $\ell$ (Placed as shown in figure).




**Solution :** $\because$ Direction of force is changing at every element. We have to make components of force and then integrate.

Net vertical force = 0.

dF = force on element = $\frac{G.dM.m}{(x^2 + a^2)}$

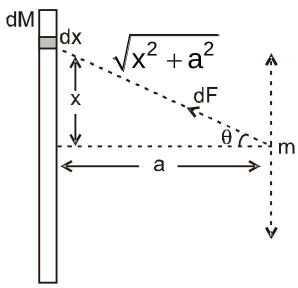
$dF_h$ = dF cos θ = force on element in horizontal direction = $\frac{G.dM.m}{(x^2 + a^2)} \cos\theta$

$$\therefore \quad F_h = \int \frac{G.M.m\cos\theta dx}{\ell(x^2 + a^2)} = \frac{G.M.m}{\ell} \int_{-\ell/2}^{\ell/2} \frac{\cos\theta \cdot dx}{(x^2 + a^2)} = \frac{GMm}{\ell a^2} \int_{-\ell/2}^{\ell/2} \frac{\cos\theta \cdot dx}{\sec^2\theta}$$

where x = a tan θ then dx = a sec²θ.dθ

$$= \frac{GMm}{\ell a} \left[\sin\theta\right]_{-\ell/2}^{\ell/2} \qquad \tan\theta = \frac{x}{a}, \text{ then } \sin\theta = \frac{x}{\sqrt{x^2 + a^2}}$$

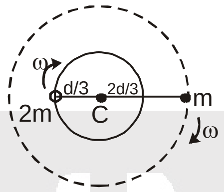
$$= \frac{GMm}{\ell a} \left[ \frac{x}{\sqrt{x^2 + a^2}} \right]_{-\ell/2}^{\ell/2} = \frac{GMm\ell}{\ell a\sqrt{\frac{\ell^2}{4} + a^2}} = \frac{GMm}{a\sqrt{\frac{\ell^2}{4} + a^2}}$$



**Problem 2.** In a double star, two stars (one of mass m and the other of 2m) distant d apart rotate about their common centre of mass. Deduce an expression of the period of revolution. Show that the ratio of their angular momentum about the centre of mass is the same as the ratio of their kinetic energies.

**Solution :** The centre of mass C will be at distances d/3 and 2d/3 from the masses 2m and m respectively. Both the stars rotate around C in their respective orbits with the same angular velocity $\omega$. The gravitational force acting on each star due to the other provides the necessary centripetal acceleration.

The gravitational force on either star is $\frac{G(2m)m}{d^2}$. If we consider the rotation of the smaller star, the centripetal force ($m\,r\,\omega^2$) is $\left[m\left(\frac{2d}{3}\right)\omega^2\right]$ and for bigger star $\left[\frac{2md\omega^2}{3}\right]$ i.e. same



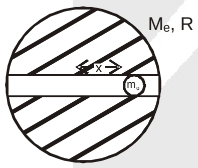

$$\therefore \frac{G(2m)m}{d^2} = m\left(\frac{2d}{3}\right)\omega^2 \quad \text{or} \quad \omega = \sqrt{\left(\frac{3Gm}{d^3}\right)}$$

Therefore, the period of revolution is given by $T = \frac{2\pi}{\omega} = 2\pi\sqrt{\left(\frac{d^3}{3Gm}\right)}$

The ratio of the angular momentum is $\frac{(I\omega)_{big}}{(I\omega)_{small}} = \frac{I_{big}}{I_{small}} = \frac{(2m)\left(\frac{d}{3}\right)^2}{m\left(\frac{2d}{3}\right)^2} = \frac{1}{2}$,

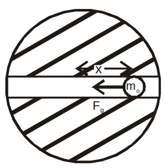
since $\omega$ is same for both. The ratio of their kinetic energies is $\frac{(\frac{1}{2}I\omega^2)_{big}}{(\frac{1}{2}I\omega^2)_{small}} = \frac{I_{big}}{I_{small}} = \frac{1}{2}$, which is the same as the ratio of their angular momentum.

**Problem 3.**




The earth can be assumed to be a uniform sphere of mass $M_e$ and radius R. A small tunnel is dug in the earth as shown. A particle of mass $m_o$ is released from radial distance x. Find the fore acting on the particle due to earth. Estimate the motion of the particle and find its time period.

**Solution :**




Magnitude of force acting on the particle = $(m_0)$ $(g_{earth})$

$= (m_0)\left(\dfrac{GM_e}{R^3}x\right)$

so $F = \left(\dfrac{GM_e m_0}{R^3}\right)x$

As this form is opposite of x so we can write

$F = -\left(\dfrac{GM_e m_0}{R^3}\right)x$

Now this form $F \propto -x$, So motion of the particle will be simple harmonic motion

$F = -\left(\dfrac{GM_e m_0}{R^3}\right)x$

$F = -Kx$

Comparing with the standard eqn. of SHM the force constant $k = \dfrac{GM_e m_0}{R^3}$

So time period of the particle.

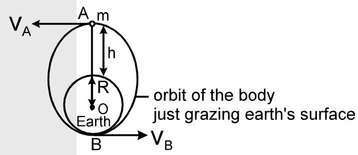
$T = 2\pi\sqrt{\dfrac{m_o}{k}}$

$T = 2\pi\sqrt{\dfrac{m_o}{\dfrac{GM_e m_0}{R^3}}}$

$T = 2\pi\sqrt{\dfrac{R^3}{GM_e}} = 2\pi\sqrt{\dfrac{R}{g}}$

Putting R = 6400km, g = 9.8 m/s$^2$ we get $T \simeq 1$hr 24 min.

**Problem 4.** For a particle projected in a transverse direction from a height h above Earth's surface, find the minimum initial velocity so that it just grazes the surface of earth in a way that path of this particle would be an ellipse with center of earth as the farther focus, point of projection as the apogee and a diametrically opposite point on earth's surface as perigee.




**Solution :** Suppose velocity of projection at point A is $v_A$ & at point B, the velocity of the particle is $v_B$. Conserving angular momentum about the centre of earth at points A and B, we get :

$mv_A(R + h) = mv_B R \Rightarrow v_B = \dfrac{v_A(R+h)}{R}$

Now applying conservation of energy at points A & B

$$\frac{-GM_e m}{R+h} + \frac{1}{2}mv_A^2 = \frac{-GM_e m}{R} + \frac{1}{2}mv_B^2$$

$$\Rightarrow \; GM_e m\left(\frac{1}{R} - \frac{1}{(R+h)}\right) = \frac{1}{2}(mv_B^2 - mv_A^2) = \frac{1}{2}mv_A^2\left[\frac{(R+h)^2}{R^2} - 1\right]$$

$$\therefore \quad v_A^2\,\frac{h(2R+h)}{R^2} = \frac{2GM_e h}{R(R+h)}$$

$$\therefore \quad v_A = \sqrt{\frac{2GM_e R}{(R+h)(2R+h)}}$$

**Problem 5.** A rocket starts vertically upward with speed $v_0$. Show that its speed v at height h is given by $v_0^2 - v^2 = \dfrac{2gh}{1+\frac{h}{R}}$, where R is the radius of the earth and g is acceleration due to gravity on earth's surface. Hence deduce an expression for maximum height reached by a rocket fired with speed 0.9 times the escape speed.

**Solution :** The gravitational potential energy of a mass m on earth's surface and that a height h is given by

$$U(R) = -\frac{GMm}{R} \text{ and } U(R+h) = -\frac{GMm}{R+h}$$

$$\therefore \quad U(R+h) - U(R) = -GMm\left(\frac{1}{R+h} - \frac{1}{R}\right) = \frac{GMmh}{(R+h)R} = \frac{mhg}{1+\frac{h}{R}} \quad [\because GM = gR^2]$$

This increase in potential energy occurs at the cost of kinetic energy which correspondingly decreases. If v is the velocity of the rocket at height h, then the decrease in kinetic energy is $\frac{1}{2}mv_0^2 - \frac{1}{2}mv^2$.

Thus, $\frac{1}{2}mv_0^2 - \frac{1}{2}mv^2 = \dfrac{mhg}{1+\frac{h}{R}}$,

or $\quad v_0^2 - v^2 = \dfrac{2gh}{1+\frac{h}{R}}$

Let $h_{max}$ be the maximum height reached by the rocket, at which its velocity has been reduced to zero. Thus, substituting v = 0 and h = $h_{max}$ in the last expression, we have

$$v_0^2 = \frac{2gh_{max}}{1+\frac{h_{max}}{R}} \text{ or } v_0^2\left(1+\frac{h_{max}}{R}\right) = 2\,gh_{max}$$

or $\quad v_0^2 = h_{max}\left(2g - \dfrac{v_0^2}{R}\right) \qquad$ or $\qquad h_{max} = \dfrac{v_0^2}{2g - \frac{v_0^2}{R}}$

Now, it is given that $v_0$ = 0.9 × escape velocity = 0.9 × $\sqrt{(2gR)}$

$$\therefore \quad h_{max} = \frac{(0.9\times0.9)2gR}{2g - \frac{(0.9\times0.9)2gR}{R}} = \frac{1.62gR}{2g-1.62R} = \frac{1.62R}{0.38} = 4.26\text{ R}$$

**Problem 6.** A test particle is moving in circular orbit in the gravitational field produced by a mass density $\rho(r) = \dfrac{K}{r^2}$. Identify the correct relation between the radius R of the particle's orbit and its period T :

(A) TR is a constant (B) $T/R^2$ is a constant (C*) T/R is a constant (D) $T^2/R^3$ is a constant

**Solution :**

$$\int_0^R\left(\frac{G}{R^2}\frac{K}{r^2}4\pi r^2dr\right)m = m\left(\frac{2\pi}{T}\right)^2\times R$$

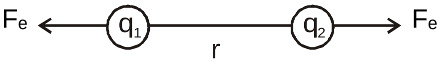
$$\frac{GK4\pi}{R^2}\times R\times m = m\left(\frac{2\pi}{T^2}\right)\times R$$

$T^2/R^2$ = constant

T/R = constant

# COMPARATIVE STUDY OF ELECTROSTATICS AND GRAVITATION

## ELECTROSTATICS

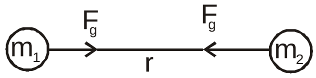



Force acting between two point charges

$F_e = \frac{kq_1q_2}{r^2}$ , $k = \frac{1}{4\pi\varepsilon_0}$

(attractive of repulsive)

## GRAVITATION




Gravitation force acting between two point masses

$F_g = \frac{G\, m_1 m_2}{r^2}$ (always attractive)

**all the formulae of gravitation are Similar to electrostatics. Simply K is replaced by G, and $q_1$, $q_2$ are replaced by $m_1$ $m_2$ .**

**Electric field (E) :-**

Electrostatic force acting on unit charge

$E = \frac{F_e}{q_o}$

If a charge $q_0$ is placed in electric field E, then force acting on the charge $\vec{F} = q_0 \vec{E}$

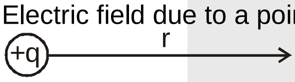
**Gravitational field (g) :-**

Gravitational force acting on unit mass $g = \frac{F_g}{m_o}$

If a mass $m_0$ is placed in a gravity field g, then force acting on the mass is $\vec{F}_g = (m_0)\ \vec{g}$

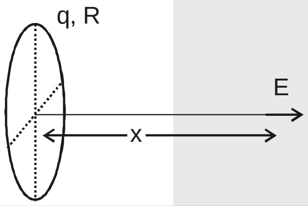
(Here force is always in the direction of $\vec{g}$)

Electric field due to a point charge



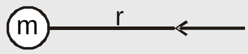

$\vec{E} = \frac{kq}{r^2}\hat{r}$

Electric field due to a uniformly charged ring




$E = \frac{Kq\,x}{\left(R^2 + x^2\right)^{3/2}}$

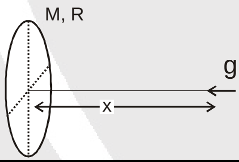
Gravitation field due point mass




$\vec{g} = -\frac{Gm}{r^2}\hat{r}$

(always towards point mass)

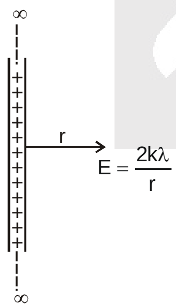
Gravitational field due to a uniform ring




$g = \frac{Gmx}{\left(R^2 + x^2\right)^{3/2}}$

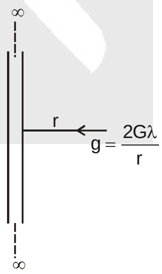
Electric field due to an infinitely long wire having

$\frac{\text{charge}}{\text{length}} = \lambda$




Gravitational field due to infinitely long wire having

$\frac{\text{mass}}{\text{length}} = \lambda$



Electric field due to a uniformly charged thin spherical shell

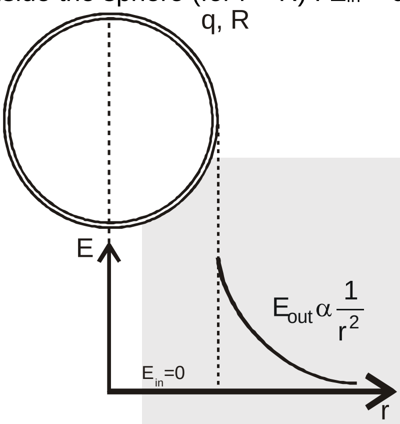
(i) Electric field outside the sphere (for r > R) :

$$\vec{E}_{out} = \frac{kq}{r^2}\hat{r} = \frac{kq}{(\text{distance from centre})^2}$$

(ii) Electric field just outside the surface

$$\vec{E}_{surface} = \frac{kq}{R^2}\hat{r}$$

(iii) E inside the sphere (for r < R) : $E_{in} = 0$




Gravitational field due to uniform thin spherical shell (hollow sphere) is

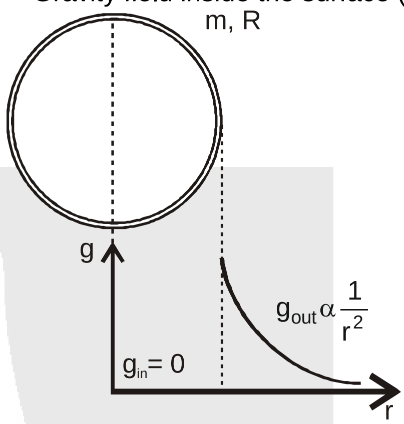
(i) Gravity field outside the sphere (for r > R):

$$\vec{g}_{out} = -\frac{Gm}{r^2}\hat{r} = \frac{Gm}{(\text{distance from center})^2}$$

(ii) gravity field just outside the surface

$$\vec{g}_{surface} = -\frac{Gm}{R^2}\hat{r}$$

(iii) Gravity field inside the surface (for r< R): $g_{in} = 0$




(figure shows magnitude of $\vec{g}$)

Electric field due to uniformly charged solid sphere

(i) Electric field outside the sphere (for r ≥ R)

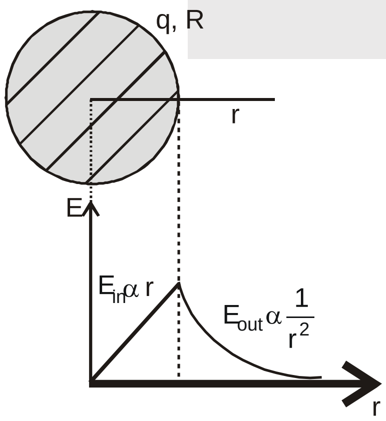
$$\vec{E}_{out} = \frac{kq}{r^2}\hat{r}$$

(ii) Electric field at the surface of the sphere (r = R)

$$\vec{E}_{surface} = \frac{kq}{R^2}\hat{r}$$

(iii) Electric field inside the sphere (for r ≤ R)

$$\vec{E}_{in} = \frac{kq}{R^3}.\vec{r}$$




Gravitational field due to uniform solid sphere

(i) Gravitational field outside the sphere (for r ≥ R)

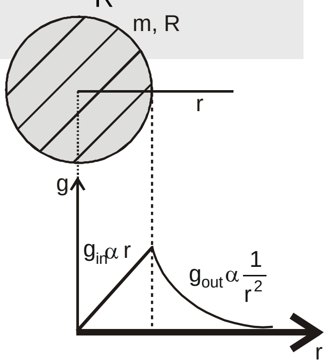
$$\vec{g}_{out} = -\frac{Gm}{r^2}\hat{r}$$

(ii) Gravitational field at the surface of the sphere (r = R)

$$\vec{g}_{surface} = -\frac{Gm}{R^2}\hat{r}$$

(iii) Gravitational field inside the sphere (for r ≤ R)

$$\vec{g}_{in} = -\frac{Gm}{R^3}.\vec{r}$$




$g_{out} \alpha \frac{1}{r^2}$

(figure shows magnitude of $\vec{g}$)

**Electrostatic potential:**

Work done by external agent to bring a unit charge from infinity to that point, slowly.

$$V_E = -\int_{r\to\infty}^{r=r} \vec{E}.\vec{dr}$$

and potential difference

$$V_B - V_A = -\int_A^B \vec{E}.\vec{dr}$$

**Gravitational potential :**

work done by external agent to bring a unit mass from infinity to that point slowly.

$$V_g = -\int_{r\to\infty}^{r=r} \vec{g}.\vec{dr}$$

and gravitational potential difference

$$V_B - V_A = -\int_A^B \vec{g}.\vec{dr}$$

Potential due a point charge

$$V = -\int_{r\to\infty}^{r=r} \vec{E}.\vec{dr}$$

$$V = -\int_{r\to\infty}^{r=r} \frac{kq}{r^2}\hat{r}.\vec{dr}$$

$$V = -\int_{r\to\infty}^{r=r} \frac{kq}{r^2}dr = \frac{kq}{r}$$

$$V = \frac{kq}{r}$$

Gravitational potential due to point mass

$$V_g = -\int_{r\to\infty}^{r=r} \vec{g}.\vec{dr}$$

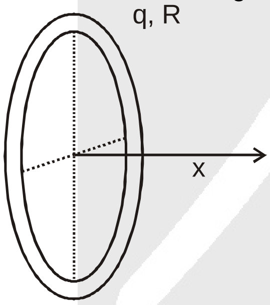
Now gravitation field due to a point mass is $\vec{g} = \frac{Gm}{r^2}(-\hat{r})$

(gravitational field is always attractive so $(-\hat{r})$ is used for direction)

$$\Rightarrow \quad V_g = -\int_{r\to\infty}^{r=r} \frac{Gm}{r^2}(-\hat{r}).\vec{dr}$$

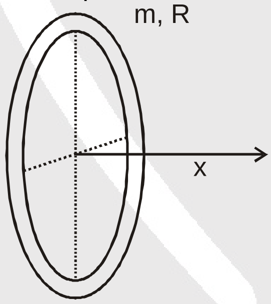
$$\Rightarrow \quad V_g = \int_{r\to\infty}^{r=r} \frac{Gm}{r^2} = -\frac{Gm}{r} \quad \Rightarrow \quad V_g = -\frac{Gm}{r}$$

Electrostatic potential due to a charged ring




$$V = \frac{kq}{\sqrt{R^2 + x^2}}$$

Gravitational potential due to a ring




$$V_g = -\frac{Gm}{\sqrt{R^2 + x^2}}$$

Electrostatic potential due to uniformly charged thin spherical shell

(i) Potential outside the shell (r > R)

$$V_{out} = \frac{kq}{r} = \frac{kq}{(\text{distance from center})}$$

(ii) Potential at the surface of the shell (r = R)

$$V_{surface} = \frac{kq}{R} = \frac{kq}{(\text{Radius of sphere})}$$

(iii) Potential inside the shell (r < R)

$$V_{in} = \frac{kq}{R} = \frac{kq}{(\text{Radius of sphere})}$$

Gravitational field due to uniform spherical shell

(i) Gravitational potential outside the shell (r > R)

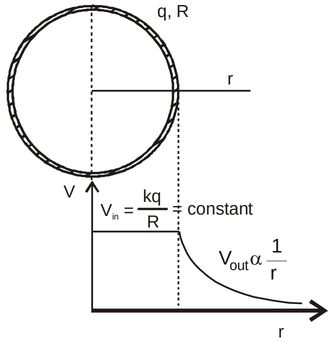
$$V_{out} = -\frac{Gm}{r} = -\frac{Gm}{(\text{distance from centre})}$$

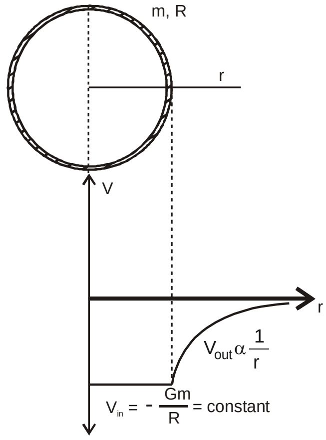
(ii) Gravitational Potential at the surface of the shell (r = R)

$$V_{surface} = -\frac{Gm}{R} = -\frac{Gm}{(\text{radius of the sphere})}$$

(iii) Gravitational Potential inside the shell (r < R)

$$V_{in} = -\frac{Gm}{R} = -\frac{Gm}{(\text{radius of the sphere})}$$




Electric potential due to a uniformly charged solid sphere

(i) Potential at outside point (r > R)

$$V_{out} = \frac{kq}{r} = \frac{kq}{(\text{distance from centre})}$$

(ii) Potential at the surface of sphere (r = R) :-

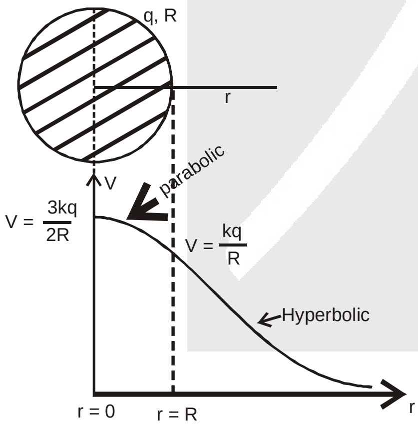
$$V_{surface} = \frac{kq}{R} = \frac{kq}{(\text{radius of the sphere})}$$

(iii) Potential at a point inside the sphere (r < R)

$$V_{in} = \frac{kq}{2R^3}(3R^2 - r^2)$$

(iv) Potential at centre (r = 0)

$$V_{centre} = \frac{3kq}{2R}$$




Gravitational potential due to a uniform solid sphere.

(i) Gravitational potential at outside point (r > R)

$$V_{out} = -\frac{Gm}{r} = -\frac{Gm}{(\text{distance from centre})}$$

(ii) Gravitational potential at the surface of sphere (r = R)

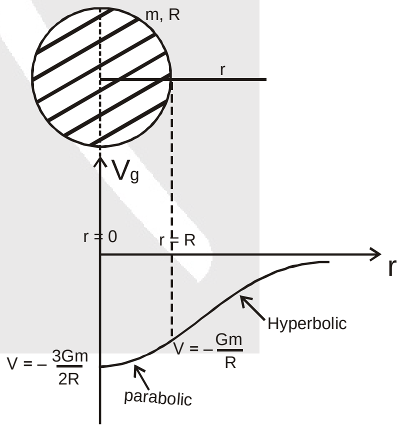
$$V_{surface} = -\frac{Gm}{R} = -\frac{Gm}{(\text{radius of the sphere})}$$

(iii) Gravitational potential at a point inside the sphere

$$V_{in} = -\frac{Gm}{2R^3}(3R^2 - r^2)$$

(iv) Gravitational potential at the centre (r = 0)

$$V_{centre} = -\frac{3Gm}{2R}$$




Field – potential relation :

$$E = -\frac{dV}{dr}$$

If V depends on x, y, z

then $\vec{E} = -\left(\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right)$

$$g = -\frac{dV_g}{dr}$$

If $V_g$ depends on x, y, z then

$$\vec{g} = -\left(\frac{\partial V_g}{\partial x}\hat{i} + \frac{\partial V_g}{\partial y}\hat{j} + \frac{\partial V_g}{\partial z}\hat{k}\right)$$

If a charge $q_o$ is placed in electrical potential V. Then electrical potential energy of the charge

$$U = q_o V$$

Self electrostatics potential energy of a uniformly charged thin spherical shell is

$$U_{self} = \frac{Kq^2}{2R}$$

Electrostatic self potential energy of a uniformly charged solid sphere is

$$U_{self} = \frac{3Kq^2}{5R}$$

If a point mass $m_o$ is placed in gravitational potential $V_g$, then the gravitational potential energy of the charge.

$$U_g = (m_o)(V_g)$$

Self gravitational potential energy of a thin uniform spherical shell is

$$(U_g)_{self} = -\frac{GM^2}{2R}$$

Self gravitational potential energy of a uniform solid sphere is

$$U_{self} = -\frac{3GM^2}{5R}$$