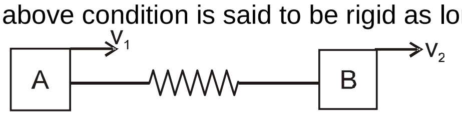
- *Imagination is more important than knowledge*
- *Everything should be made as simple as possible, but not simpler.* - *Albert Einstein*

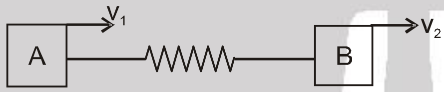
RIGID BODY DYNAMICS

1. RIGID BODY :

Rigid body is defined as a system of particles in which distance between each pair of particles remains constant (with respect to time). Remember, rigid body is a mathematical concept and any system which satisfies the above condition is said to be rigid as long as it satisfies it.

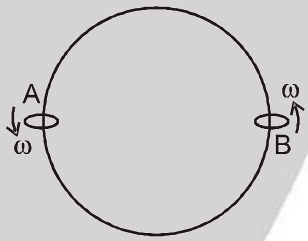
☞ ($v_1 = v_2$)


System behaves as a rigid body

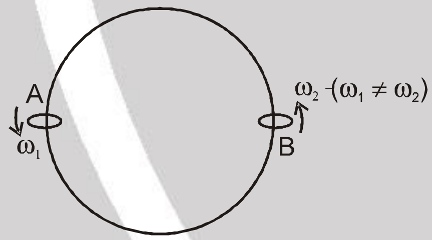
($v_1 \neq v_2$)


System behaves as a non-rigid body

☞ A & B are beads which move on a circular fixed ring

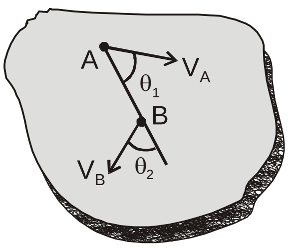


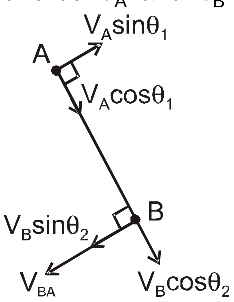
A + B is a rigid body system
but A + B + ring is non-rigid system



A + B is non-rigid system

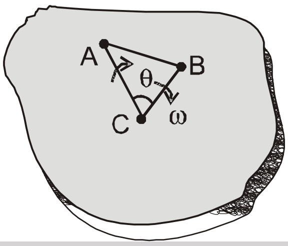
- If a system is rigid, since there is no change in the distance between any pair of particles of the system, shape and size of system remains constant. Hence we intuitively feel that while a stone or cricket ball are rigid bodies, a balloon or elastic string is non rigid.
  But any of the above system is rigid as long as relative distance does not change, whether it is a cricket ball or a balloon. But at the moment when the bat hits the cricket ball or if the balloon is squeezed, relative distance changes and now the system behaves like a non-rigid system.
- For every pair of particles in a rigid body, there is no velocity of separation or approach between the particles. i.e. any relative motion of a point B on a rigid body with respect to another point A on the rigid body will be perpendicular to line joining A to B, hence with respect to any particle A of a rigid body the motion of any other particle B of that rigid body is circular motion.
  Let velocities of A and B with respect ground be $\vec{V}_A$ and $\vec{V}_B$ respectively in the figure below.



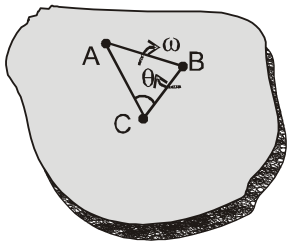


If the above body is rigid $V_A \cos\theta_1 = V_B \cos\theta_2$ (velocity of approach / separation is zero)
V_{BA} = relative velocity of B with respect to A.
$V_{BA} = V_A \sin\theta_1 + V_B \sin\theta_2$ (which is perpendicular to line AB)
B will appear to move in a circle to an observer fixed at A.

- W.r.t. any point of the rigid body the angular velocity of all other points of the that rigid body is same.
- Suppose A, B, C is a rigid system hence during any motion sides AB, BC and CA must rotate through the same angle. Hence all the sides rotate by the same rate.



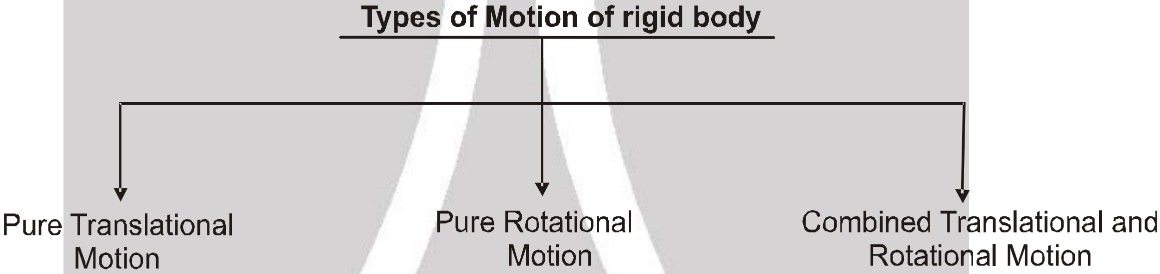
(i)



(ii)

From figure (i) angular velocity of A and B w.r.t. C is ω,

From figure (ii) angular velocity of A and C w.r.t. B is ω,



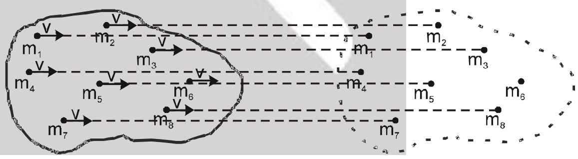
I. Pure Translational Motion :

A body is said to be in pure translational motion, if the displacement of each particle of the system is same during any time interval. During such a motion, all the particles have same displacement $(\vec{s})$, velocity $(\vec{v})$ and acceleration $(\vec{a})$ at an instant.

Consider a system of n particle of mass m_1, m_2, m_3, ... m_n under going pure translation then from above definition of translational motion

$\vec{a}_1 = \vec{a}_2 = \vec{a}_3 = ...\vec{a}_n = \vec{a}$ (say)

and $\vec{v}_1 = \vec{v}_2 = \vec{v}_3 = ...\vec{v}_n = \vec{v}$ (say)



From Newton's laws for a system.

$$\vec{F}_{ext} = m_1\vec{a}_1 + m_2\vec{a}_2 + m_3\vec{a}_3 + ...$$

$$\vec{F}_{ext} = M\vec{a}$$

Where M = Total mass of the body

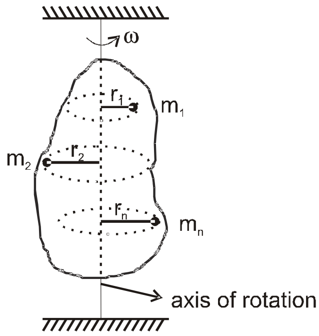
$$\vec{P} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + ...$$

$$\vec{P} = M\vec{v}$$

Total Kinetic Energy of body = $\frac{1}{2}$ $m_1v_1^2 + m_2v_2^2 +$ .... = Mv^2

II. Pure Rotational Motion :

Figure shows a rigid body of arbitrary shape in rotation about a fixed axis, called the axis of rotation. Every point of the body moves in a circle whose center lies on the axis of rotation, and every point moves through the same angle during a particular time interval. Such a motion is called pure rotation.



We know that each particle has same angular velocity (since the body is rigid.)

so, $v_1 = \omega r_1$, $v_2 = \omega r_2$, $v_3 = \omega r_3$ ...... $v_n = \omega r_n$

Total Kinetic Energy = $\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 +$ ....

$= \frac{1}{2} [m_1 r_1^2 + m_2 r_2^2 + \text{..........................}]\, \omega^2$

$= \frac{1}{2} I\omega^2$ Where $I = m_1 r_1^2 + m_2 r_2^2 +$ .... (is called moment of inertia)

ω = angular speed of body.

III. Combined Translational and Rotational Motion :

A body is said to be in combined translation and rotational motion if all point in the body rotates about an axis of rotation and the axis of rotation moves with respect to the ground. Any general motion of a rigid body can be viewed as a combined translational and rotational motion.

Solved Examples

Example 1. A body is moving down into a well through a rope passing over a fixed pulley of radius 10 cm. Assume that there is no slipping between rope & pulley. Calculate the angular velocity and angular acceleration of the pulley at an instant when the body is going down at a speed of 20 cm/s and has an acceleration of 4.0 m/s^2.

Solution : Since the rope does not slip on the pulley, the linear speed v of the rim of the pulley is same as the speed of the body.

The angular velocity of the pulley is then $\omega = v/r = \frac{20\text{cm/s}}{10\text{cm}} = 2$ rad/s

and the angular acceleration of the pulley is $\alpha = a/r = \frac{4.0\ \text{m/s}^2}{10\ \text{cm}} = 40$ rad/s^2.

Example 2. A disc rotates with a uniform angular acceleration of 2.0 rad/s^2 about its axis. If the disc starts from rest, how many revolutions will it make in the first 10 seconds?

Solution : The angular displacement in the first 10 seconds is given by

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 = \frac{1}{2}(2.0\ \text{rad/s}^2)\ (10\text{s})^2 = 100\ \text{rad.}$$

As the wheel turns by 2π radian in each revolution, the number of revolutions in 10s is

$$n = \frac{100}{2\pi} = 16.$$

Example 3. The wheel of a motor, accelerated uniformly from rest, rotates through 5 radian during the first second. Calculate the angle rotated during the next second.

Solution : As the angular acceleration is constant, we have

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 = \frac{1}{2}\alpha t^2$$

Thus, 5 rad = $\frac{1}{2}\alpha\ (1\text{s})^2$

$\alpha = 10$ rad/s^2 or $\alpha = 10$ rad/s^2

The angle rotated during the first two seconds is = $\frac{1}{2} \times (10\ \text{rad/s}^2)\ (2\text{s})^2 = 20$ rad

Thus, the angle rotated during the 2nd second is 20 rad – 5 rad = 15 rad

Example 4. Starting from rest, a fan takes four seconds to attain the maximum speed of 400 rpm (revolution per minute). Assuming uniform acceleration, calculate the time taken by the fan in attaining half the maximum speed.

Solution : Let the angular acceleration be α. According to the question,

$400 \text{ rev/min} = 0 + \alpha\, 4$ ....(i)

Let t be the time taken in attaining the speed of 200 rev/min which is half the maximum.

Then, $200 \text{ rev/min} = 0 + \alpha t$ ....(ii)

Dividing (i) by (ii), we get,

$t = 2$ sec.

Example 5. The motor of an engine is rotating about its axis with an angular velocity of 120 rev/minute. It comes to rest in 10 s, after being switched off the engine. Assuming uniform angular deceleration, find the number of revolutions made by it before coming to rest.

Solution : The initial angular velocity = 120 rev/minute = (4π) rad/s.

Final angular velocity = 0.

Time interval = 10 s.

Let the angular acceleration be α. Using the equation $\omega = \omega_0 + \alpha t$, we obtain

$\alpha = (-4\pi/10)$ rad/s^2

The angle rotated by the motor during this motion is

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 = \left(4\pi\frac{\text{rad}}{\text{s}}\right)(10\text{s}) - \frac{1}{2}\left(\frac{4\pi}{10}\frac{\text{rad}}{\text{s}^2}\right)(10\text{s})^2$$

$= 20\pi$ rad = 10 revolutions.

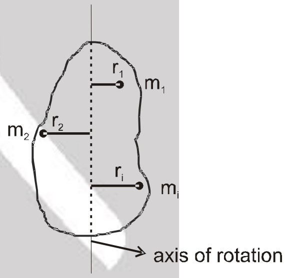
Hence the motor rotates through 10 revolutions before coming to rest.

2. MOMENT OF INERTIA (I) ABOUT AN AXIS :

(i) Moment of inertia of a system of n particles about an axis is defined as :

$I = m_1 r_1^2 + m_2 r_2^2 + ... + m_n r_n^2$

i.e. $I = \sum_{i=1}^{n} m_i r_i^2$



where, r_i = It is perpendicular distance of mass m_i from axis of rotation

SI units of Moment of Inertia is Kgm^2.

Moment of inertia is a scalar positive quantity.

(ii) For a continuous system :

$I = \int r^2 (dm)$

where dm = mass of a small element

r = perpendicular distance of the mass element dm from the axis

Moment of Inertia depends on :

(i) density of the material of body
(ii) shape & size of body
(iii) axis of rotation

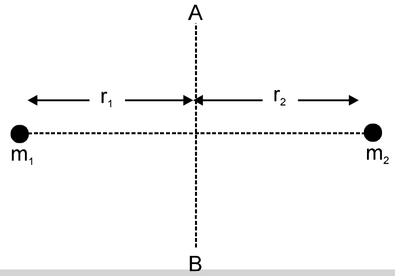
In totality we can say that it depends upon distribution of mass relative to axis of rotation.

Note : Moment of inertia does not change if the mass :

(i) is shifted parallel to the axis of the rotation because r_i does not change.
(ii) is rotated about axis of rotation in a circular path because r_i does not change.

Solved Examples

Example 6. Two particles having masses m_1 & m_2 are situated in a plane perpendicular to line AB at a distance of r_1 and r_2 respectively as shown.



(i) Find the moment of inertia of the system about axis AB ?
(ii) Find the moment of inertia of the system about an axis passing through m_1 and perpendicular to the line joining m_1 and m_2 ?
(iii) Find the the moment of inertia of the system about an axis passing through m_1 and m_2?
(iv) Find moment of inertia about axis passing through centre of mass and perpendicular to line joining m_1 and m_2

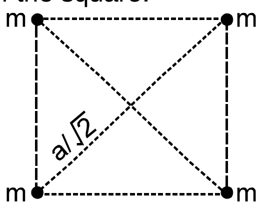
Solution : (i) Moment of inertia of particle on left is $I_1 = m_1r_1^2$.
Moment of Inertia of particle on right is $I_2 = m_2r_2^2$.
Moment of Inertia of the system about AB is $I = I_1 + I_2 = m_1r_2^2 + m_2r_2^2$
(ii) Moment of inertia of particle on left is $I_1 = 0$
Moment of Inertia of particle on right is $I_2 = m_2(r_1 + r_2)^2$.
Moment of Inertia of the system about AB is $I = I_1 + I_2 = 0 + m_2 (r_1 + r_2)^2$
(iii) Moment of inertia of particle on left is $I_1 = 0$
Moment of Inertia of particle on right is $I_2 = 0$
Moment of Inertia of the system about AB is $I = I_1 + I_2 = 0 + 0$
(iv) Centre of mass of system $r_{CM} = m_2\left(\frac{r_1+r_2}{m_1+m_2}\right)$ = Distance of centre mass from mass m_1

Distance of centre of mass from mass $m_2 = m_1\left(\frac{r_1+r_2}{m_1+m_2}\right)$

So moment of inertia about centre of mass = $I_{cm} = m_1\left(m_2\frac{r_1+r_2}{m_1+m_2}\right)^2 + m_2\left(m_1\frac{r_1+r_2}{m_1+m_2}\right)^2$

$$I_{CM} = \frac{m_1m_2}{m_1+m_2}(r_1 + r_2)^2$$

Example 7. Four particles each of mass m are kept at the four corners of a square of edge a. Find the moment of inertia of the system about a line perpendicular to the plane of the square and passing through the centre of the square.

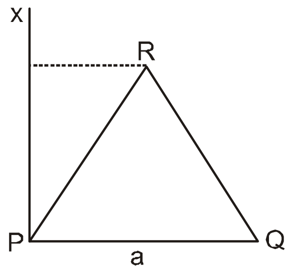


Solution : The perpendicular distance of every particle from the given line is $a/\sqrt{2}$. The moment of inertia of one particle is, therefore, $m(a/\sqrt{2})^2 = \frac{1}{2}ma^2$. The moment of inertia of the system is, therefore,

$$4 \times \frac{1}{2}ma^2 = 2ma^2$$

Example 8. Three particles, each of mass m, are situated at the vertices of an equilateral triangle PQR of side a as shown in figure. Calculate the moment of inertia of the system about

(i) The line PX perpendicular to PQ in the plane of PQR.
(ii) One of the sides of the triangle PQR
(iii) About an axis passing through the centroid and perpendicular to plane of the triangle PQR.



Solution : (i) Perpendicular distance of P from PX = 0
Perpendicular distance of Q from PX = a
Perpendicular distance of R from PX = a/2
Thus, the moment of inertia of the particle at P = 0, of the particle at Q - ma^2, and of the particle at R = $m(a/2)^2$. The moment of inertia of the three-particle system about PX is

$$0 + ma^2 + m(a/2)^2 = \frac{5ma^2}{4}$$

Note that the particles on the axis do not contribute to the moment of inertia.

(ii) Moment of inertia about the side PR = mass of particle Q × square of perpendicular distance of Q from side PR, $I_{PR} = m\left(\frac{\sqrt{3}}{2}a\right)^2 = \frac{3ma^2}{4}$

(iii) Distance of centroid from all the particle is $\frac{a}{\sqrt{3}}$, so moment of inertia about an axis and passing through the centroic perpendicular plane of triangle PQR = $I_R = 3m\left(\frac{a}{\sqrt{3}}\right)^2 = ma^2$

Example 9. Calculate the moment of inertia of a ring having mass M, radius R and having uniform mass distribution about an axis passing through the centre of ring and perpendicular to the plane of ring ?

Solution : $I = \int (dm)\ r^2$

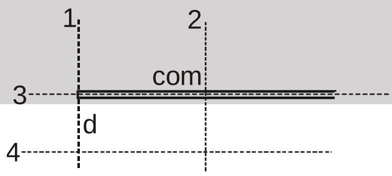
Because each element is equally distanced from the axis so r = R

$= R^2 \int dm = MR^2$

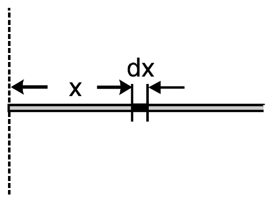
$I = MR^2$

(Note : Answer will remain same even if the mass is nonuniformly distributed because $\int dm = M$ always.)

Example 10. Calculate the moment of inertia of a uniform rod of mass M and length ℓ about an axis 1, 2, 3 and 4.



Solution $(I_1) = \int (dm)\ r^2 = \int_0^{\ell} \left(\frac{M}{\ell}dx\right) x^2 = \frac{M\ell^2}{3}$

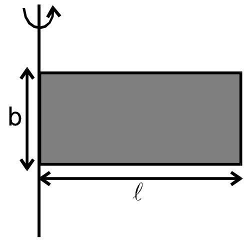


$(I_2) = \int (dm)\ r^2 = \int_{-\ell/2}^{\ell/2} \left(\frac{M}{\ell}dx\right) x^2 = \frac{M\ell^2}{12}$

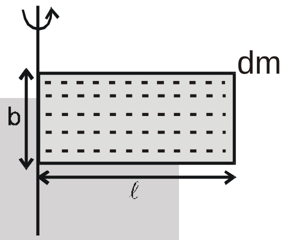
$(I_3) = 0$ (axis 3 passing through the axis of rod)

$(I_4) = d^2 \int (dm) = Md^2$

Example 11. Determined the moment of inertia of a uniform rectangular plate of mass, side 'b' and 'ℓ'' about an axis passing through the edge 'b' and in the plane of plate.

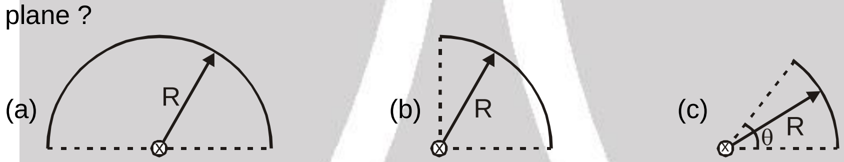


Solution : Each section of dm mass rod in the rectangular plate has moment of inertia about an axis passing through edge 'b' $dI = \frac{dm\ell^2}{3}$



So $I = \int dI = \frac{\ell^2}{3}\int dm = \frac{M\ell^2}{3}$

Example 12. Find out the moment of Inertia of figures shown each having mass M, radius R and having uniform mass distribution about an axis passing through the centre and perpendicular to the plane ?



Solution : MR^2 (infact M.I. of any part of mass M of a ring of radius R about axis passing through geometrical centre and perpendicular to the plane of the ring is = MR^2)

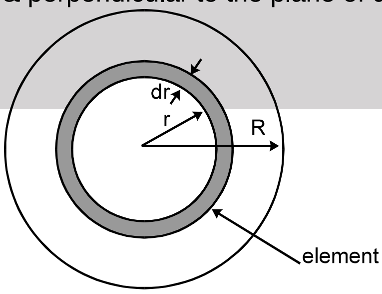
(iii) Moment of inertia of a large object can be calculated by integrating M.I.of an element of the object**:**

$I = \int dI_{element}$ where dI = moment of inertia of a small element.

Element chosen should be such that : either perpendicular distance of axis from each point of the element is same or the moment of inertia of the element about the axis of rotation is known.

Solved Examples

Example 13. Determine the moment of Inertia of a uniform disc having mass M, radius R about an axis passing through centre & perpendicular to the plane of disc ?



Solution : $I = \int dI_{ring}$

element - ring $dI = dmr^2$

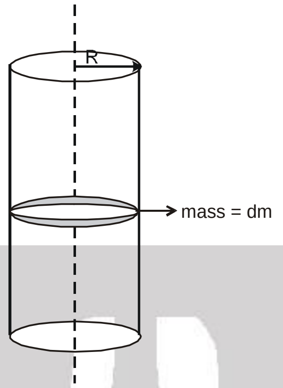
$dm = \frac{M}{\pi R^2} 2\pi r dr$ (here we have used the uniform mass distribution)

$\therefore \quad I = \int_0^R \frac{M}{\pi R^2}.(2\pi r dr).r^2 \quad \Rightarrow \quad I = \frac{MR^2}{2}$

Example 14. Calculate the moment of inertia of a uniform hollow cylinder of mass M, radius R and length ℓ about its axis.

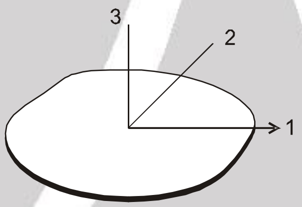
Solution : Moment of inertia of a uniform hollow cylinder is

$$I = \int (dm)R^2 = mR^2$$



3. TWO IMPORTANT THEOREMS ON MOMENT OF INERTIA

(i) Perpendicular Axis Theorem **[Only applicable to plane laminar bodies (i.e. for 2 dimensional objects only)].**



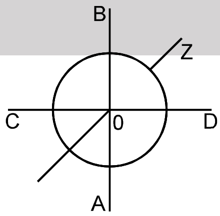
Body is in 1-2 plane

If axis 1 & 2 are in the plane of the body and perpendicular to each other.
Axis 3 in perpendicular to plane of 1 & 2.
Then, $I_3 = I_1 + I_2$

☞ The point of intersection of the three axis need not be center of mass, it can be any point in the plane of body which lies on the body or even outside it.

Solved Examples

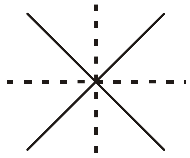
Example 15. Calculate the moment of inertia of a uniform disc of mass M and radius R about a diameter.



Solution : Let AB and CD be two mutually perpendicular diameters of the disc. Take them as X and Y-axes and the line perpendicular to the plane of the disc through the centre as the Z-axis. The moment of inertia of the ring about the Z-axis is $I = \frac{1}{2}MR^2$. As the disc is uniform, all of its diameters are equivalent and so $I_x = I_y$, From perpendicular axes theorem,

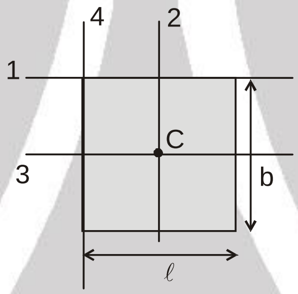
$I_z = I_x + I_y$. Hence $I_x = \frac{I_z}{2} = \frac{MR^2}{4}$

Example 16. Two uniform identical rods each of mass M and lengthare joined to form a cross as shown in figure. Find the moment of inertia of the cross about a bisector as shown dotted in the figure.



Solution : Consider the line perpendicular to the plane of the figure through the centre of the cross. The moment of inertia of each rod about this line is $\frac{M\ell^2}{12}$ and hence the moment of inertia of the cross is $\frac{M\ell^2}{6}$. The moment of inertia of the cross about the two bisector are equal by symmetry and according to the theorem of perpendicular axes, the moment of inertia of the cross about the bisector is $\frac{M\ell^2}{12}$.

Example 17. In the figure shown find moment of inertia of a plate having mass M, length ℓ and width b about axis 1,2,3 and 4. Assume that mass is uniformly distributed.



Solution : Moment of inertia of the plate about axis 1 (by taking rods perpendicular to axis 1)

$I_1 = Mb^2 / 3$

Moment of inertia of the plate about axis 2 (by taking rods perpendicular to axis 2)

$I_2 = M\ell^2 / 12$

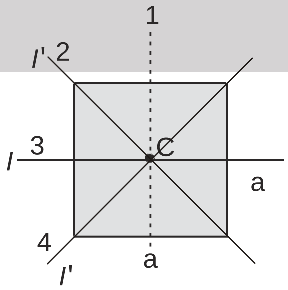
Moment of inertia of the plate about axis 3 (by taking rods perpendicular to axis 3)

$I_3 = Mb^2 / 12$

Moment of inertia of the plate about axis 4 (by taking rods perpendicular to axis 4)

$I_4 = M\ell^2 / 3$

Example 18. In the figure shown find the moment of inertia of square plate having mass m and sides a. About an axis 2 passing through point C (centre of mass) and in the plane of plate.



Solution : Using perpendicular axis theorems $I_C = I_4 + I_2 = 2I'$

Using perpendicular theorems $I_C = I_3 + I_1 = I + I = 2I$

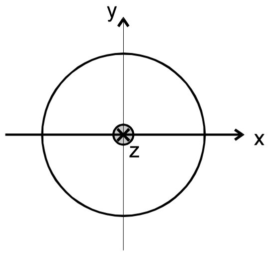
$2I' = 2I$

$I' = I$

$IC = 2I = \frac{ma^2}{6} \quad \Rightarrow \quad I' = \frac{ma^2}{12}$

Example 19. Find the moment of Inertia of a uniform disc of mass M and radius R about a diameter.

Solution : Consider x & y two mutually perpendicular diameters of the ring.



$I_x + I_y = I_z$

$I_x = I_y$ (due to symmetry)

$I_z = \dfrac{MR^2}{2}$

$I_x = I_y = \dfrac{MR^2}{4}$

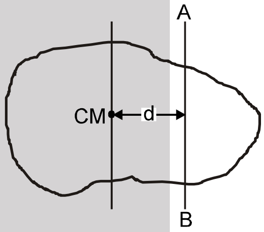
(ii) Parallel Axis Theorem (Applicable to planer as well as 3 dimensional objects):

If I_{AB} = Moment of Inertia of the object about axis AB

I_{cm} = Moment of Inertia of the object about an axis passing through centre of mass and parallel to axis AB

M = Total mass of object

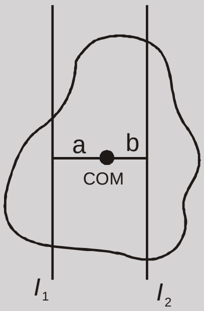
d = perpendicular distance between axis AB about which moment of Inertia is to be calculated & the one passing through the centre of mass and parallel to it.



$I_{AB} = I_{cm} + Md^2$

Solved Examples

Example 20. Find out relation between I_1 and I_2. I_1 and I_2 moment of inertia of a rigid body mass m about an axis as shown in figure.



Solution : Using parallel axis theorem

$I_1 = I_C + ma^2$ .....(1)

$I_2 = I_C + mb^2$ .....(2)

From (1) and (2) ; $I_1 - I_2 = m(a^2 - b^2)$

Example 21. Find the moment of inertia of a uniform sphere of mass m and radius R about a tangent if the spheres (i) solid (ii) hollow

Solution : (i) Using parallel axis theorem $I = I_{CM} + md^2$

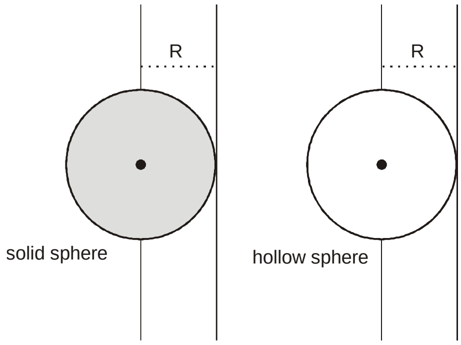
for solid sphere $I_{CM} = \dfrac{2}{5}mR^2$, d = R

$I = \dfrac{7}{5}mR^2$

(ii) Using parallel axis theorem $I = I_{CM} + md^2$

for hollow sphere $I_{CM} = \dfrac{2}{3}mR^2$, d = R

$I = \dfrac{5}{3}mR^2$



Example 22. Calculate the moment of inertia of a hollow cylinder of mass M and radius R about a line parallel to the axis of the cylinder and on the surface of the cylinder.

Solution : The moment of inertia of the cylinder about its axis = MR^2.

Using parallel axes theorem, $I = I_0 + MR^2 = MR^2 + MR^2 = 2\ MR^2$.

Similarly, the moment of inertia of a hollow sphere about a tangent is

$$\frac{2}{3}MR^2 + MR^2 = \frac{5}{3}MR^2$$

Example 23. Find out the moment of inertia of a semi circular disc about an axis passing through its centre of mass and perpendicular to the plane?

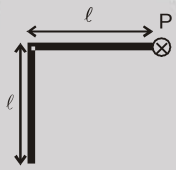
Solution : Moment of inertia of a semi circular disc about an axis passing through centre and perpendicular to plane of disc, $I = \frac{MR^2}{2}$

Using parallel axis theorem $I = I_{CM} + Md^2$, d is the perpendicular distance between two parallel axis passing through centre C and COM.

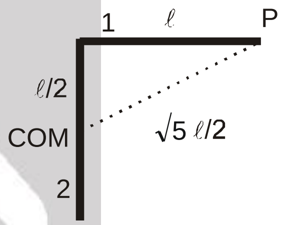
$$I = \frac{MR^2}{2},\ d = \frac{4R}{3\pi} \qquad \Rightarrow \qquad \frac{MR^2}{2} = I_{CM} + M\left(\frac{4R}{3\pi}\right)^2$$

$$I_{CM} = \left[\frac{MR^2}{2} - M\left(\frac{4R}{3\pi}\right)^2\right]$$

Example 24. Find the moment of inertia of the two uniform joint rods having mass m each about point P as shown in figure. Using parallel axis theorem.



Solution : Moment of inertia of rod 1 about axis P , $I_1 = \frac{m\ell^2}{3}$



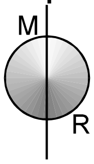
Moment of inertia of rod 2 about axis P, $I_2 = \frac{m\ell^2}{12} + m\left(\sqrt{5}\frac{\ell}{2}\right)^2$

So moment of inertia of a system about axis P ,

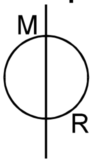
$$I = I_1 + I_2 = \frac{m\ell^2}{3} + \frac{m\ell^2}{12} + m\left(\sqrt{5}\frac{\ell}{2}\right)^2$$

$$I = \frac{m\ell^2}{3}$$

List of some useful formule :

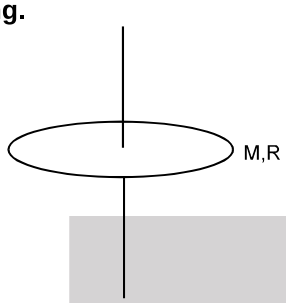
Object	Moment of Inertia
Solid Sphere 	$\frac{2}{5}MR^2$ (Uniform)

Hollow Sphere



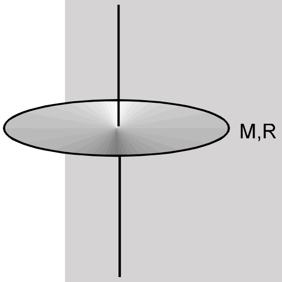
$\frac{2}{3}MR^2$ (Uniform)

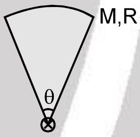
Ring.



MR^2 (Uniform or Non Uniform)

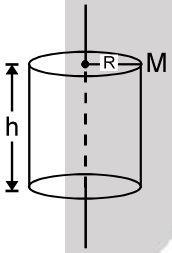
Disc

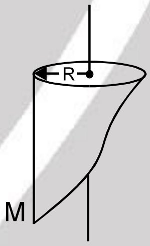




$\frac{MR^2}{2}$ (Uniform)

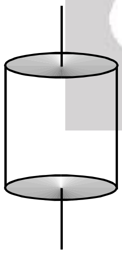
Hollow cylinder





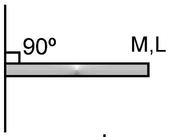
MR^2 (Uniform or Non Uniform)

Solid cylinder

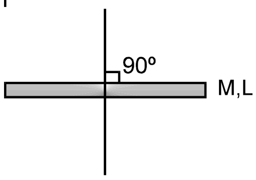


$\frac{MR^2}{2}$ (Uniform)

Thin rod

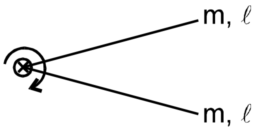


$\frac{ML^2}{3}$ (Uniform)



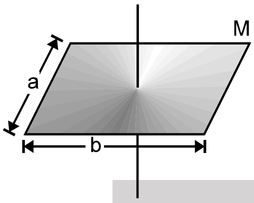
$\frac{ML^2}{12}$ (Uniform)

Two thin rod



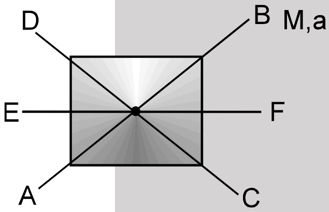
$\frac{2m\ell^2}{3}$ (Uniform)

Rectangular Plate



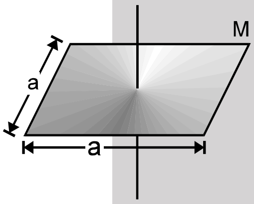
$I = \frac{M(a^2 + b^2)}{12}$ (Uniform)

Square Plate



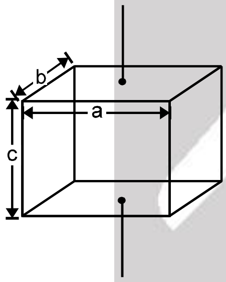
$I_{AB} = I_{CD} = I_{DF} = \frac{Ma^2}{12}$ (Uniform)

Square Plate



$\frac{Ma^2}{6}$ (Uniform)

Cuboid



$\frac{M(a^2 + b^2)}{12}$ (Uniform)

4. RADIUS OF GYRATION :

As a measure of the way in which the mass of rigid body is distributed with respect to the axis of rotation, we define a new parameter, the radius of gyration (K). It is related to the moment of inertia and total mass of the body.

$I = MK^2$

where I = Moment of Inertia of a body

M = Mass of a body

K = Radius of gyration

$K = \sqrt{\frac{I}{M}}$

☞ Length K is the geometrical property of the body and axis of rotation.

S.I. Unit of K is meter.

Solved Examples

Example 25. Find the radius of gyration of a solid uniform sphere of radius R about its tangent.

Solution : $I = \frac{2}{5}mR^2 + mR^2 = \frac{7}{5}mR^2 = mK^2 \quad \Rightarrow K = \sqrt{\frac{7}{5}}R$

Example 26. Find the radius of gyration of a hollow uniform sphere of radius R about its tangent.

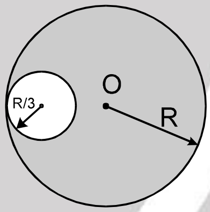
Solution : Moment of inertia of a hollow sphere about a tangent, $I = \frac{5}{3}MR^2$

$MK^2 = \frac{5}{3}MR^2 \quad \Rightarrow \quad K = \sqrt{\frac{5}{3}}R$

5. MOMENT OF INERTIA OF BODIES WITH CUT :

Solved Examples

Example 27 A uniform disc of radius R has a round disc of radius R/3 cut as shown in Fig. .The mass of the remaining (shaded) portion of the disc equals M. Find the moment of inertia of such a disc relative to the axis passing through geometrical centre of original disc and perpendicular to the plane of the disc.



Solution : Let the mass per unit area of the material of disc be σ. Now the empty space can be considered as having density –σ and σ.

Now $I_0 = I_\sigma + I_{-\sigma}$

$I_\sigma = (\sigma \pi R^2)R^2/2$ = M.I. of σ about o

$I_{-\sigma} = \frac{-\sigma\pi(R/3)^2(R/3)^2}{2} + [-\sigma\pi(R/3)^2]\,(2R/3)^2$ = M.I. of –σ about o

$\therefore \quad I_0 = \frac{4}{9}MR^2$ **Ans.**

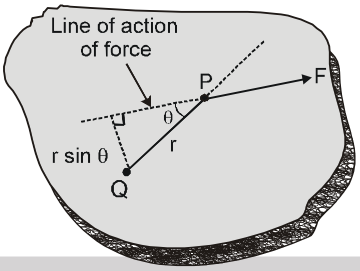
Example 28. Find the moment of inertia of a uniform disc of radius R_1 having an empty symmetric annular region of radius R_2 in between, about an axis passing through geometrical centre and perpendicular to the disc.

Solution : $\rho = \frac{M}{\pi\ (R_1^2 - R_2^2)} \quad \Rightarrow \quad I = \rho \times \left(\frac{\pi R_1^4 - \pi R_2^4}{2}\right)$

$I = \frac{M\left(R_1^{\,2} + R_2^{\,2}\right)}{2}$ **Ans.**

6. TORQUE :

Torque represents the capability of a force to produce change in the rotational motion of the body.



6.1 Torque about a point :

Torque of force $\vec{F}$ about a point $\vec{\tau} = \vec{r} \times \vec{F}$

Where $\vec{F}$ = force applied

P = point of application of force

Q = Point about which we want to calculate the torque.

$\vec{r}$ = position vector of the point of application of force w.r.t. the point about which we want to determine the torque.

$|\vec{\tau}| = r\,F\,\sin\theta = r_\perp F = rF_\perp$

Where θ = angle between the direction of force and the position vector of P wrt. Q.

$r_\perp = r \sin\theta$ = perpendicular distance of line of action of force from point Q ,it is also called force arm.

$F_\perp = F \sin\theta$ = component of $\vec{F}$ perpendicular to $\vec{r}$

SI unit of torque is Nm

☞ Torque is a vector quantity and its direction is determined using right hand thumb rule and its always perpendicular to the plane of rotation of the body.

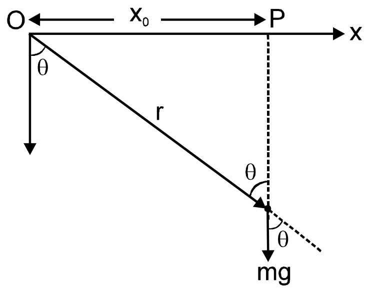
Solved Examples

Example 29. A particle of mass M is released in vertical plane from a point P at x = x_0 on the x-axis it falls vertically along the y-axis. Find the torque τ acting on the particle at a time t about origin ?

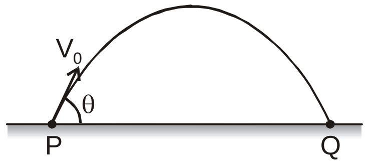
Solution : Torque is produced by the force of gravity.

$$\vec{\tau} = r\,F\,\sin\theta\ \hat{k}$$

or $\tau = r_\perp F = x_0 mg = r\,mg\,\dfrac{x_0}{r} = mgx_0\ \hat{k}$



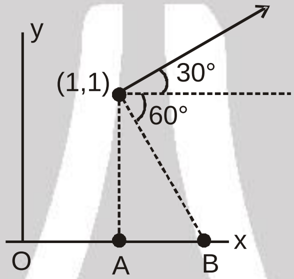
Example 30. A particle having mass m is projected with a velocity v_0 from a point P on a horizontal ground making an angle θ with horizontal. Find out the torque about the point of projection acting on the particle when it is at its maximum height ?



Solution : $\tau = rF\sin\theta = \frac{R}{2}\ mg = \frac{v_0^{\ 2}\sin 2\theta}{2g}\ mg$

$\tau = \frac{mv_0^{\ 2}\sin 2\theta}{2}$

Example 31. Find the torque about point O and A.



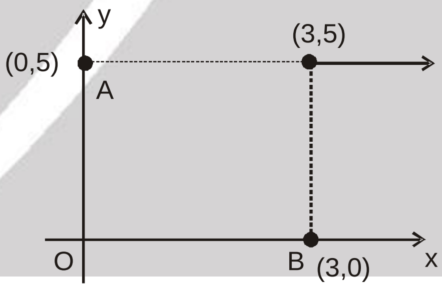
Solution : Torque about point O, $\vec{\tau} = \vec{r}_0 \times \vec{F}$, $\vec{r}_0 = \hat{i} + \hat{j}$, $\vec{F} = 5\sqrt{3}\hat{i} + 5\hat{j}$

$\vec{\tau} = (\hat{i} + \hat{j}) \times (5\sqrt{3}\hat{i} + 5\hat{j}) = 5(1 - \sqrt{3})\hat{k}$

Torque about point A , $\vec{\tau} = \vec{r}_a \times \vec{F}$, $\vec{r}_a = \hat{j}$, $\vec{F} = 5\sqrt{3}\hat{i} + 5\hat{j}$

$\vec{\tau} = \hat{j} \times (5\sqrt{3}\hat{i} + 5\hat{j}) = 5(-\sqrt{3})\hat{k}$

Example 32. Find out torque about point A, O and B.



Solution : Torque about point A , $\vec{\tau}_A = \vec{r}_A \times \vec{F}$, $\vec{r}_A = 3\hat{i}$, $\vec{F} = 10\hat{i}$

$\vec{\tau}_A = 3\hat{i} \times 10\hat{i} = 0$

Torque about point B , $\vec{\tau}_B = \vec{r}_B \times \vec{F}$, $\vec{r}_B = 5\hat{j}$, $\vec{F} = 10\hat{i}$

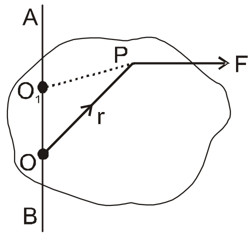
$\vec{\tau}_B = 5\hat{j} \times 10\hat{i} = -50\hat{k}$

Torque about point O , $\vec{\tau}_O = \vec{r}_O \times \vec{F}$, $\vec{r}_O = 3\hat{i} + 5\hat{j}$, $\vec{F} = 10\hat{i}$

$\vec{\tau}_O = (3\hat{i} + 5\hat{j}) \times 10\hat{i} = -50\hat{k}$

6.2 Torque about an axis :

The torque of a force $\vec{F}$ about an axis AB is defined as the component of torque of $\vec{F}$ about any point O on the axis AB, along the axis AB.



In the given figure torque of $\vec{F}$ about O is $\vec{\tau}_0 = \vec{r} \times \vec{F}$

The torque of $\vec{F}$ about AB, τ_{AB} is component of $\vec{\tau}_0$ along line AB.

There are four cases of torque of a force about an axis.:

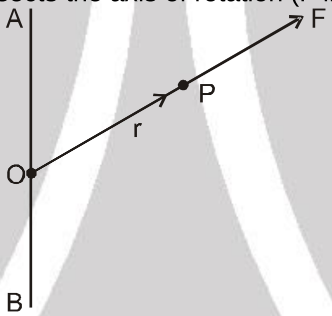
Case I : Force is parallel to the axis of rotation, $\vec{F} \parallel \overrightarrow{AB}$

AB is the axis of rotation about which torque is required

$\vec{r} \times \vec{F}$ is perpendicular to $\vec{F}$, but $\vec{F} \parallel \overrightarrow{AB}$, hence $\vec{r} \times \vec{F}$ is perpendicular to $\overrightarrow{AB}$.

The component of $\vec{r} \times \vec{F}$ along $\overrightarrow{AB}$ is, therefore, zero.

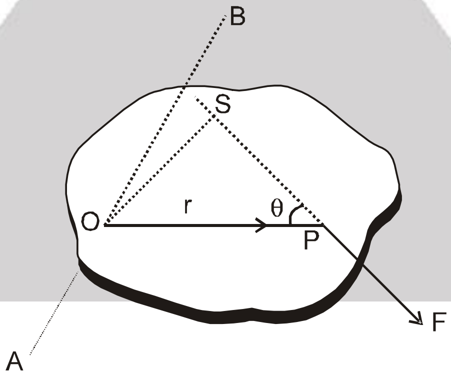
Case II : The line of force intersects the axis of rotation (F intersect AB)



$\vec{F}$ intersects AB along $\vec{r}$ then $\vec{F}$ and $\vec{r}$ are along the same line. The torque about O is $\vec{r} \times \vec{F} = 0$.

Hence component this torque along line AB is also zero.

Case III : $\vec{F}$ perpendicular to $\overrightarrow{AB}$ but $\vec{F}$ and AB do not intersect.



In the three dimensions, two lines may be perpendicular without intersecting each other. Two nonparallel and nonintersecting lines are called skew lines.

Figure shows the plane through the point of application of force P that is perpendicular to the axis of rotation AB. Suppose the plane intersects the axis at the point O. The force F is in this plane (since F is perpendicular to AB). Taking the origin at O,

Torque = $\vec{r} \times \vec{F} = \overrightarrow{OP} \times \vec{F}$.

Thus, torque = rF sin θ = F(OS)

where OS is the perpendicular from O to the line of action of the force $\vec{F}$. The line OS is also perpendicular to the axis of rotation. It is thus the length of the common perpendicular to the force and the axis of rotation.

The direction of $\vec{\tau} = \overrightarrow{OP} \times \vec{F}$ is along the axis AB because $\overrightarrow{AB} \perp \overrightarrow{OP}$ and $\overrightarrow{AB} \perp \vec{F}$. The torque about AB is, therefore, equal to the magnitude of $\vec{\tau}$ that is F.(OS).

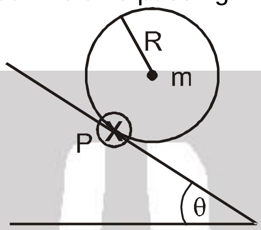
☞ Thus, the torque of F about AB = magnitude of the force F × length of the common perpendicular to the force and the axis. The common perpendicular OS is called the lever arm or moment arm of this torque.

Case IV : $\vec{F}$ and $\overrightarrow{AB}$ are skew but not perpendicular.

Here we resolve $\vec{F}$ into two components, one is parallel to axis and other is perpendicular to axis. Torque of the parallel part is zero and that of the perpendicular part may be found, by using the result of **case (III)**.

Solved Examples

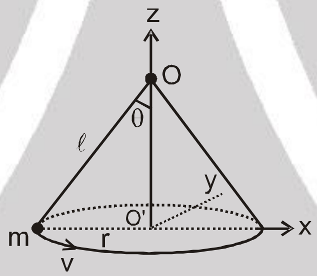
Example 33. Find the torque of weight about the axis passing through point P.



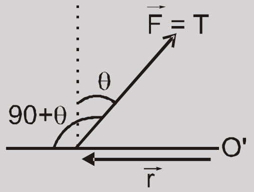
Solution : $\vec{\tau} = \vec{r} \times \vec{F}$, $\vec{r} = R$, $\vec{F} = mg\sin\theta$

r and F both are at perpendicular so torque about point P = mgRsinθ

Example 34. A bob of mass m is suspended at point O by string of length ℓ. Bob is moving in a horizontal circle find out (i) torque of gravity and tension about point O and O'. (ii) Net torque about axis OO' .



Solution :



(i) Torque about point O

Torque of tension (T), $\tau_{ten} = 0$ (tension is passing through point O)

Torque of gravity $\tau_{mg} = \ell mg\sin\theta$

Torque about point O'

Torque of gravity $\tau_{mg} = mgr$ $\quad r = \ell\sin\theta$

Torque of tension $\tau_{mg} = \ell mg\sin\theta$ (along negative $\hat{j}$)

Torque of tension $\tau_{ten} = Tr\sin(90+\theta)$ $\quad (T\cos\theta = mg)$

$\tau_{ten} = Tr\cos\theta$

$\tau_{ten} = \dfrac{mg}{\cos\theta}(\ell\sin\theta)\cos\theta = mg\ell\sin\theta$ (along positive $\hat{j}$)

(ii) Torque about axis OO'

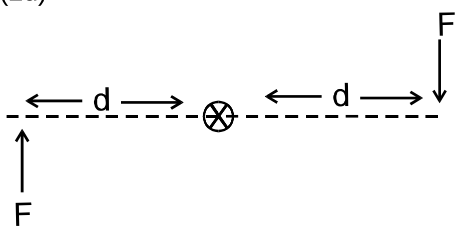
Torque of gravity about axis OO' $\tau_{mg} = 0$ (force mg parallel to axis OO')

Torque of tension about axis OO' $\tau_{ten} = 0$ (force T is passing through the axis OO')

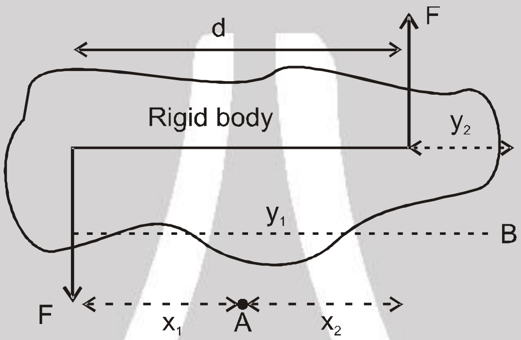
Net torque about axis OO' $\tau_{net} = 0$

6.3 Force Couple :

A pair of forces each of same magnitude and acting in opposite direction is called a force couple. Torque due to couple = Magnitude of one force × distance between their lines of action.
Magnitude of torque = τ = F (2d)



- A couple does not exert a net force on an object even though it exerts a torque.
- Net torque due to a force couple is same about any point.

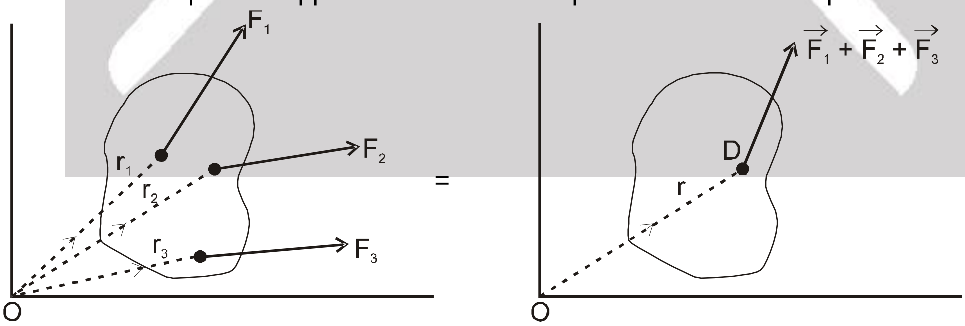


Torque about A = $x_1F + x_2F$ = $F(x_1 + x_2)$ = Fd
Torque about B = $y_1F - y_2F$ = $F(y_1 - y_2)$ = Fd

- If net force acting on a system is zero, torque is same about any point.
- A consequence is that, if F_{net} = 0 and τ_{net} = 0 about one point, then τ_{net} = 0 about any point.

6.4 Point of Application of Force :

Point of Application of force is the point at which, if net force is assumed to be acting, then it will produce same translational as well as rotational effect, as was produced earlier.
We can also define point of application of force as a point about which torque of all the forces is zero.

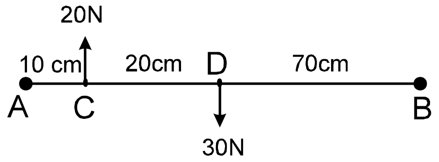


Consider three forces $\vec{F}_1, \vec{F}_2, \vec{F}_3$ acting on a body if D is point of application of force then torque of $\vec{F}_1 + \vec{F}_2 + \vec{F}_3$ acting at a point D about O is same as the original torque about O

$$\left[\vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \vec{r}_3 \times \vec{F}_3\right] = \vec{r} \times (\vec{F}_1 + \vec{F}_2 + \vec{F}_3)$$

Solved Examples

Example 35. Determine the point of application of force, when forces of 20 N & 30 N are acting on the rod as shown in figure.

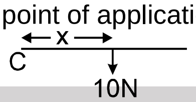


Solution : Net force acting on the rod $F_{rel} = 10N$

Nett torque acting on the rod about point C

$\tau_c = (20 \times 0) + (30 \times 20) = 600$ clockwise

Let the point of application be at a distance x from point C



$600 = 10\,x \Rightarrow \quad x = 60$ cm

$\therefore$ 70 cm from A is point of Application

Note : (i) Point of application of gravitational force is known as the centre of gravity.
(ii) Centre of gravity coincides with the centre of mass if value of is assumed to be constant.
(iii) Concept of point of application of force is imaginary, as in some cases it can lie outside the body.

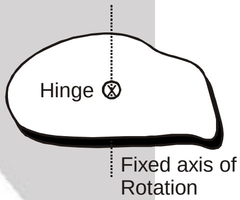
6.5 Rotation about a fixed axis :

If I_{Hinge} = moment of inertia about the axis of rotation (since this axis passes through the hinge, hence the name I_{Hinge}).

$\vec{\tau}_{ext})$ = resultant external torque acting on the body about axis of rotation

α = angular acceleration of the body.

$$\vec{\tau}_{ext})_{Hinge} = I_{Hinge}\,\vec{\alpha}$$



Rotational Kinetic Energy = $\frac{1}{2}.I.\omega^2$

$$\vec{P} = M\vec{v}_{CM}$$

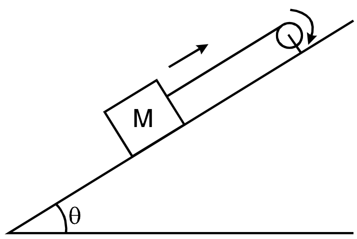
$$\vec{F}_{external} = M\ \vec{a}_{CM}$$

Net external force acting on the body has two component tangential and centripetal.

$\Rightarrow F_C = ma_C = m\frac{v^2}{r_{CM}} = m\omega^2 r_{CM}$ $\qquad \Rightarrow F_t = ma_t = m\alpha\, r_{CM}$

Solved Examples

Example 36. A pulley having radius r and moment of inertia I about its axis is fixed at the top of an inclined plane of inclination θ as shown in figure. A string is wrapped round the pulley and its free end supports a block of mass m which can slide on the plane. Initially, the pulley is rotating at a speed ω_0 in a direction such that the block slides up the plane. Calculate the distance moved by the block before stopping ?



Solution : Suppose the deceleration of the block is a. The linear deceleration of the rim of the pulley is also a. The angular deceleration of the pulley is α = a/r. If the tension in the string is T, the equations of motion are as follows :

$mg \sin\theta - T = ma$ and $Tr = I\alpha = Ia/r$.

Eliminating T from these equations,

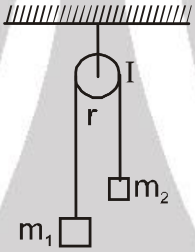
$$mg \sin\theta - I\frac{a}{r^2} = ma$$

giving, $a = \dfrac{mg \ \ r^2 \sin\theta}{I + mr^2}$

The initial velocity of the block up the incline is v = ω_0 r. Thus, the distance moved by the block before stopping is

$$x = \frac{v^2}{2a} = \frac{\omega_0^2 r^2 (I + mr^2)}{2m \ \ r^2 \sin\theta} = \frac{(I + mr^2)\omega_0^2}{2m \ \ g\sin\theta}$$

Example 37. The pulley shown in figure has a moment of inertia I about its axis and its radius is r. Calculate the magnitude of the acceleration of the two blocks. Assume that the string is light and does not slip on the pulley.



Solution : Suppose the tension in the left string is T_1 and that in the right string in T_2. Suppose the block of mass m_1 goes down with an acceleration α and the other block moves up with the same acceleration. This is also the tangential acceleration of the rim of the wheel as the string does not slip over the rim. The angular acceleration of the wheel is, therefore, α = a/r. The equations of motion for the mass m_1, the mass m_2 and the pulley are as follows :

$m_1g - T_1 = m_1a$ ......(i)

$T_2 - m_2g = m_2a$ ......(ii)

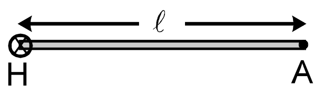
$T_1r - T_2r = I\alpha = I\alpha /r$ ......(iii)

Putting T_1 and T_2 from (i) and (ii) into (iii),

$$[(m_1g - a) - m_2 (g + a)]\, r = I\frac{a}{r}$$

which gives $a = \dfrac{(m_1 - m_2)gr^3}{I + (m_1 + m_2)r^2}$.

Example 38. A uniform rod of mass m and length ℓ can rotate in vertical plane about a smooth horizontal axis hinged at point H.



(i) Find angular acceleration α of the rod just after it is released from initial horizontal position from rest ?

(ii) Calculate the acceleration (tangential and radial) of point A at this moment.

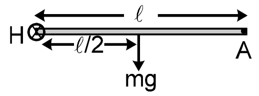
(iii) Calculate net hinge force acting at this moment.

(iv) Find α and ω when rod becomes vertical.

(v) Find hinge force when rod become vertical.

Solution : (i) $\tau_H = I_H \ \alpha$

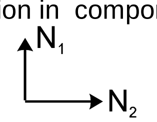
$mg.\ \frac{\ell}{2} = \frac{m\ell^2}{3}\alpha \quad \Rightarrow \quad \alpha = \frac{3g}{2\ell}$



(ii) $a_{tA} = \alpha\ell = \frac{3g}{2\ell}.\ell \quad = \frac{3g}{2}$

$a_{CA} = \omega^2 r = 0.\ell = 0$ $\quad$ ($\omega = 0$ just after release)

(iii) Suppose hinge exerts normal reaction in component form as shown



In vertical direction

$F_{ext} = ma_{CM}$

$\Rightarrow \quad mg - N_1 = m.\ \frac{3g}{4}$ $\quad$ (we get the value of a_{CM} from previous example)

$\Rightarrow \quad N_1 = \frac{mg}{4}$

In horizontal direction

$F_{ext} = ma_{CM} \Rightarrow \quad N_2 = 0$ $\quad$ (a_{CM} in horizontal = 0 as $\omega = 0$ just after release).

(vi) Torque = 0 when rod becomes vertical.

so $\alpha = 0$

using energy conservation $\quad \frac{mg\ell}{2} = \frac{1}{2}I\omega^2 \quad \left(I = \frac{m\ell^2}{3}\right)$

$\omega = \sqrt{\frac{3g}{\ell}}$

(v) When rod becomes vertical

$\alpha = 0,\ \omega = \sqrt{\frac{3g}{\ell}}$

$F_H - mg = \frac{m\omega^2\ell}{2}$

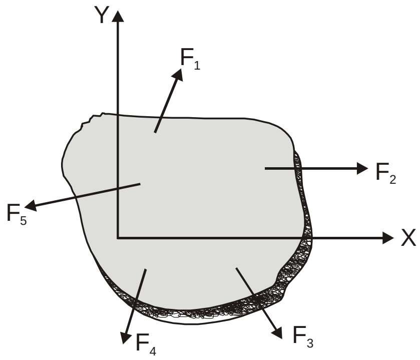
$F_H = \frac{5mg}{2}$ **Ans.**

7. EQUILIBRIUM

A system is in mechanical equilibrium if it is in translational as well as rotational equilibrium.

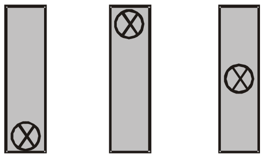
For this : $\quad \overrightarrow{F_{net}} = 0$

$\overrightarrow{\tau_{net}} = 0$ (about every point)



From (6.3), if $\overrightarrow{F_{net}} = 0$ then $\overrightarrow{\tau_{net}}$ is same about every point

Hence necessary and sufficient condition for equilibrium is $\overrightarrow{F_{net}} = 0$, $\overrightarrow{\tau_{net}} = 0$ about any one point, which we can choose as per our convenience. ($\overrightarrow{\tau_{net}}$ will automatically be zero about every point)

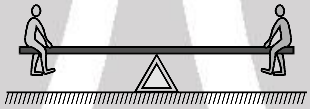
☞ 

unstable equilibrium | stable equilibrium | Neutral equilibrium

The equilibrium of a body is called **stable** if the body tries to regain its equilibrium position after being slightly displaced and released. It is called **unstable** if it gets further displaced after being slightly displaced and released. If it can stay in equilibrium even after being slightly displaced and released, it is said to be in **neutral** equilibrium.

Solved Examples

Example 39. Two boys weighing 20 kg and 25 kg are trying to balance a seesaw of total length 4m, with the fulcrum at the centre. If one of the boys is sitting at an end, where should the other sit ?



Solution : It is clear that the 20 kg kid should sit at the end and the 25 kg kid should sit closer to the centre. Suppose his distance from the centre is x. As the boys are in equilibrium, the normal force between a boy and the seesaw equals the weight of that boy. Considering the rotational equilibrium of the seesaw, the torque of the forces acting on it should add to zero. The forces are

(a) (25 kg) g downward by the 25 kg boy,

(b) (20 kg) g downward by the 20 kg boy,

(c) weight of the seesaw and

(d) the normal force by the fulcrum.

Taking torques about the fulcrum, (25 kg)g x = (20 kg)g (2 m) or x = 1.6 m.

Example 40. A uniform rod of mass m = 15 kg leans against a smooth vertical wall making an angle θ = 37° with horizontal. The other ends rests on a rough horizontal floor. Calculate the normal force and the friction force that the floor exerts on the rod. [Take g = 10 m/s^2]

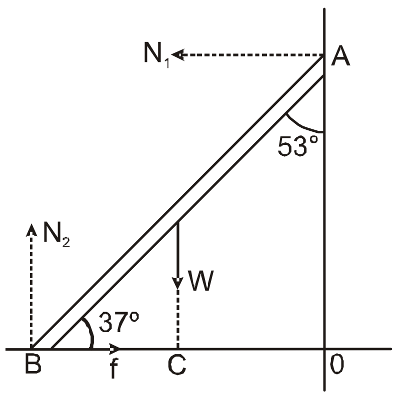
Solution : The forces acting on the rod are shown in figure. They are

(a) Its weight W,

(b) normal force N_1 by the vertical wall,

(c) normal force N_2 by the floor and

(d) frictional force f by the floor.



Taking horizontal and vertical components,

$N_1 = f$ ....(i)

and $N_2 = mg$ ....(ii)

Taking torque about B,

$N_1(AO) = mg(CB)$

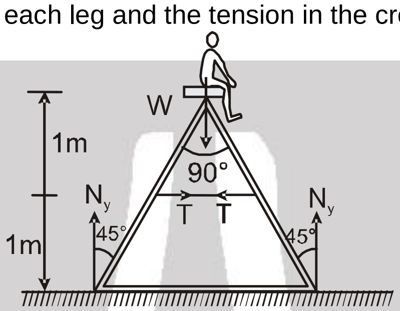
or, $N_1(AB)\cos\theta = mg\,\frac{AB}{2}\sin\theta$ or $N_1\frac{3}{5} = \frac{W}{2}\frac{4}{5}$

or, $N_1 = \frac{2}{3}W$ ....(iii)

The normal force by the floor is $N_2 = W = (15\text{ kg})(10\text{ m/s}^2) = 150$ N.

The frictional force is $f = N_1 = \frac{2}{3}W = 100$ N.

Example 41. The ladder shown in figure has negligible mass and rests on a smooth floor. A crossbar connects the two legs of the ladder at the centre as shown in figure. The angle between the two legs is 90°. The person sitting on the ladder has a mass of 60 kg. Calculate the contact forces exerted by the floor on each leg and the tension in the crossbar. [Take g = 10m/s^2]



Solution : The forces acting on different parts are shown in figure. Consider the vertical equilibrium of "the ladder plus the person" system. The forces acting on this system are its weight (60 kg)g and the contact force $N_y + N_y = 2N_y$ due to the floor. Thus

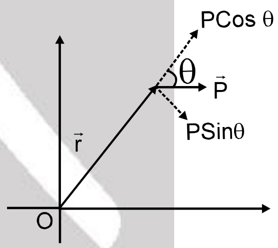
$2N_y = (60\text{ kg})\,g$ or $N_y = (30\text{ kg})(10\text{ m/s}^2) = 300$ N.

Next consider the equilibrium of the left leg of the ladder. Taking torques of the forces acting on it about the upper end,

$N_y(2m)\tan 45° = T(1\text{ m})$ or $T = N_y\,2 = (300\text{ N}) \times 2 = 600$ N.

8. ANGULAR MOMENTUM ($\vec{L}$)

8. 1. Angular momentum of a particle about a point.



$\vec{L} = \vec{r} \times \vec{P}$ ⇒ $L = rp\sin\theta$

or $|\vec{L}| = r_\perp \times P$ or $|\vec{L}| = P_\perp \times r$

Where $\vec{P}$ = momentum of particle

$\vec{r}$ = position of vector of particle with respect to point O about which angular momentum is to be calculated.

θ = angle between vectors $\vec{r}$ & $\vec{P}$

$r_\perp$ = perpendicular distance of line of motion of particle from point O.

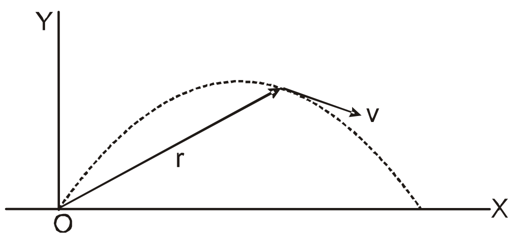
$P_\perp$ = component of momentum perpendicular to $\vec{r}$.

SI unit of angular momentum is kgm^2/sec.

Solved Examples

Example 42. A particle of mass m is projected at time t = 0 from a point O with a speed u at an angle of 45° to the horizontal. Calculate the magnitude and the direction of the angular momentum of the particle about the point O at time t = u/g.

Solution : Let us take the origin at P, X-axis along the horizontal and Y-axis along the vertically upward direction as shown in figure. For horizontal motion during the time 0 to t,



$v_x = u\cos 45° = u/\sqrt{2}$

and $x = v_x t = \frac{u}{\sqrt{2}}.\frac{u}{g} = \frac{u^2}{\sqrt{2}g}$

For vertical motion, $v_y = u \sin 45° = \frac{u}{\sqrt{2}} - u = \frac{(1-\sqrt{2})}{\sqrt{2}} u$ and $y = (u \sin 45°)\, t - \frac{1}{2} gt^2$

$$= \frac{u^2}{\sqrt{2}g} - \frac{u^2}{2g} = \frac{u^2}{2g}(\sqrt{2} - 1).$$

The angular momentum of the particle at time t about the origin is $L = \vec{r} \times \vec{p} = m\vec{r} \times \vec{v}$

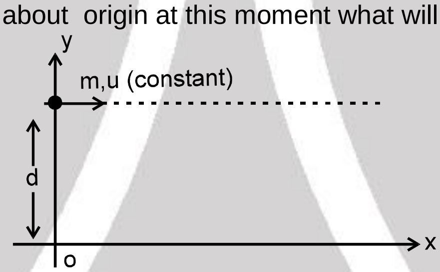
$$= m(\hat{i}\,x + \hat{j}\,y) \times (\hat{i}\,v_x + \hat{j}\,v_y) = m(\hat{k}\,xv_y - \hat{k}\,yv_x)$$

$$= m\hat{k}\left[\left(\frac{u^2}{\sqrt{2}\ g}\right)\frac{u}{\sqrt{2}}(1-\sqrt{2}) - \frac{u^2}{2g}(\sqrt{2}-1)\frac{u^2}{\sqrt{2}}\right] = -\hat{k}\ \frac{mu^3}{2\sqrt{2}g}\ .$$

Thus, the angular momentum of the particle is $\frac{mu^3}{2\sqrt{2}g}$ in the negative Z-direction i.e., perpendicular to the plane of motion, going into the plane.

Solved Examples

Example 43. A particle of mass 'm' starts moving from point (o,d) with a constant velocity $u\hat{i}$. Find out its angular momentum about origin at this moment what will be the answer at the later time?

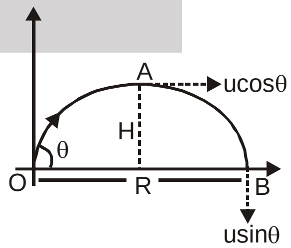


Solution : $\vec{L} = -m\,d\,u\,\hat{k}$.

Example 44. A particle of mass 'm' is projected on horizontal ground with an initial velocity of u making an angle θ with horizontal. Find out the angular momentum of particle about the point of projection when .
(i) it just starts its motion
(ii) it is at highest point of path.
(iii) it just strikes the ground.

Answer : (i) O ; (ii) $mu \cos\theta \frac{u^2 \sin^2\theta}{2g}$; (iii) $mu \sin\theta \frac{u^2 \sin 2\theta}{g}$

Solution : (i) Angular momentum about point O is zero.



(ii) Angular momentum about point A.

$\vec{L} = \vec{r} \times \vec{p}$

$L = H \times mu \cos\theta$

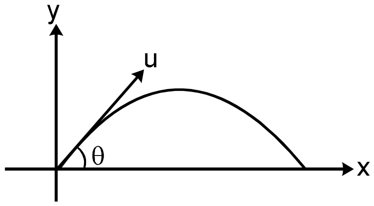
$L = mu \cos\theta \frac{u^2 \sin^2\theta}{2g}$ **Ans.**

(iii) Angular momentum about point B.

$L = R \times mu \sin\theta$

$mu \sin\theta \frac{u^2 \sin 2\theta}{g}$ **Ans.**

Example 45. A particle of mass'm' is projected on horizontal ground with an initial velocity of u making an angle θ with horizontal. Find out the angular momentum at any time t of particle p about :
(i) y axis (ii) z-axis



Solution : (i) velocity components are parallel to the y-axis. so, L = 0

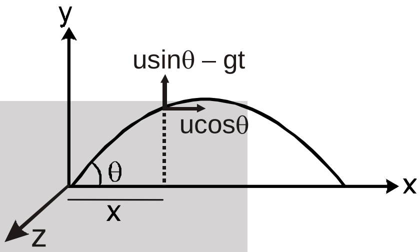
(ii) $\tau = \frac{dL}{dt}$ $-1/2$ mu cos θ. gt^2

$-mgx = \frac{dL}{dt} - mgx\, dt = dL$

$\int_0^t -mgx \quad dt = \int_0^L dL$

angular momentum about the z-axis is :

$L = -1/2$ mu cos θ. gt^2 **Ans.**



8.2 Angular momentum of a rigid body rotating about fixed axis :

Angular momentum of a rigid body about the fixed axis AB is

$L_{AB} = L_1 + L_2 + L_3 + \ldots\ldots + L_n$

$L_1 = m_1 r_1 \omega r_1$, $L_2 = m_2 r_2 \omega r_2$, $L_3 = m_3 r_3 \omega r_3$, $L_n = m_n r_n \omega r_n$

$L_{AB} = m_1 r_1 \omega r_1 + m_2 r_2 \omega r_2 + m_3 r_3 \omega r_3 \ldots\ldots + m_n r_n \omega r_n$

$$L_{AB} = \sum_{n=1}^{n=n} m_n \quad (r_n)^2 \times \omega \qquad \left[\sum_{n=1}^{n=n} m_n \quad (r_n)^2 = I_H \right]$$

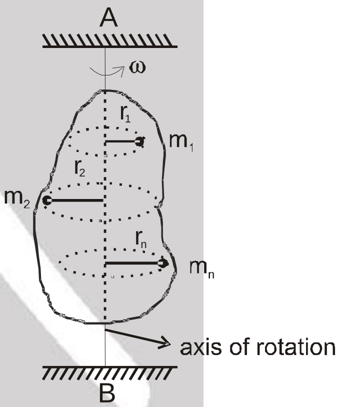
$L_{AB} = I_H\, \omega$

$L_H = I_H\, \omega$

L_H = Angular momentum of object about axis of rotation.

I_H = Moment of Inertia of rigid, body about axis of rotation.

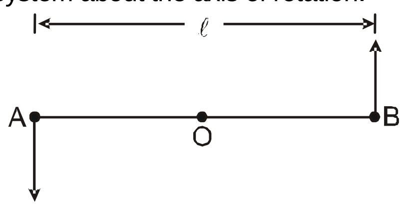
ω = angular velocity of the object.



Solved Examples

Example 46. Two particles balls A and B, each of mass m, are attached rigidly to the ends of a light rod of length ℓ. The system rotates about the perpendicular bisector of the rod at an angular speed ω. Calculate the angular momentum of the individual particles and of the system about the axis of rotation.

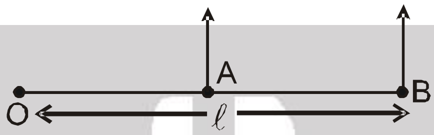
Solution : Consider the situation shown in figure. The velocity of the particle A with respect to the centre O is $v = \frac{\omega\ell}{2}$. The angular momentum of the particle with respect to the axis is $L_1 = mvr = m\left(\frac{\omega\ \ell}{2}\right)\left(\frac{\ell}{2}\right) = \frac{1}{4} m\omega\ell^2$. The same the angular momentum L_2 of the second particle. The angular momentum of the system is equal to sum of these two angular momentum i.e., $L = 1/2\ m\omega\ell^2$.



Example 47. Two small balls of mass m each are attached to a light rod of length ℓ, one at its centre and the other at a free end. The rod is fixed at the other end and is rotated in horizontal plane at an angular speed ω. Calculate the angular momentum of the ball at the end with respect to the ball at the centre.

Solution : The situation is shown in figure. The velocity of the ball A with respect to the fixed end O is $v_A = \omega(\ell/2)$ and that of B with respect to O is $v_B = \omega\ell$. Hence the velocity of B with respect to A is $v_B - v_A = \omega(\ell/2)$. The angular momentum of B with respect to A is, therefore,

$$L = mvr = m\omega\left(\frac{\ell}{2}\right)\frac{\ell}{2} = \frac{1}{4}m\omega\ell^2$$



along the direction perpendicular to the plane of rotation.

Example 48. A uniform circular ring of mass 400 g and radius 10 cm is rotated about one of its diameter at an angular speed of 20 rad/s. Find the kinetic energy of the ring and its angular momentum about the axis of rotation.

Solution : The moment of inertia of the circular ring about its diameter is

$$I = \frac{1}{2}Mr^2 = \frac{1}{2}(0.400\text{ kg})(0.10\text{ m})^2 = 2 \times 10^{-3}\text{ kg-m}^2.$$

The kinetic energy is $K = I\omega^2 = (2 \times 10^{-3}\text{ kg - m}^2)(400\text{ rad}^2/\text{s}^2) = 0.4$ J

and the angular momentum about the axis of rotation is

$L = I\omega = (2 \times 10^{-3}\text{ kg-m}^2)(20\text{ rad/s}) = 0.04\text{ kg-m}^2/\text{s} = 0.04$ J-s.

8.3 Conservation of Angular Momentum

Newton's 2nd law in rotation : $\vec{\tau} = \dfrac{d\vec{L}}{dt}$

where $\vec{\tau}$ and $\vec{L}$ are about the same axis.

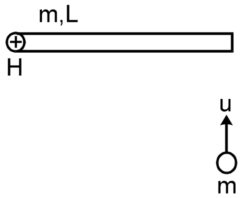
☞ Angular momentum of a particle or a system remains constant if $\tau_{ext} = 0$ about the axis of rotation. Even if net angular momentum is not constant, one of its component of an angular momentum about an axis remains constant if component of torque about that axis is zero

Impulse of Torque : $\int \tau dt = \Delta J$

$\Delta J \to$ Charge in angular momentum.

Solved Example

Example 49. A uniform rod of mass m and length ℓ can rotate freely on a smooth horizontal plane about a vertical axis hinged at point H. A point mass having same mass m coming with an initial speed u perpendicular to the rod, strikes the rod in-elastically at its free end. Find out the angular velocity of the rod just after collision ?

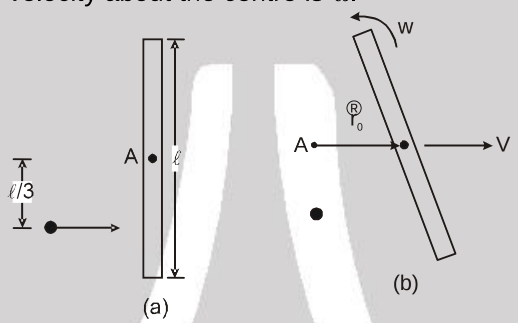


Solution : Angular momentum is conserved about H because no external force is present in horizontal plane which is producing torque about H.

$$\text{mul} = \left(\frac{m\ell^2}{3} + m\ell^2\right)\omega \qquad \Rightarrow \qquad \omega = \frac{3a}{4\ell}$$

Example 50. A uniform rod of mass m_1 and length ℓ lies on a frictionless horizontal plane. A particle of mass m_2 moving at a speed v_0 perpendicular to the length of the rod strikes it at a distance $\ell/3$ from the centre and stops after the collision. Calculate (a) the velocity of the centre of the rod and (b) the angular velocity of the rod about its centre just after the collision.

Solution : The situation is shown in figure. Consider the rod and the particle together as the system. As there is no external resultant force, the linear momentum of the system will remains constant. Also there is no resultant external torque on the system and so the angular momentum of the system about the any line will remain constant. Suppose the velocity of the centre of the rod is V and the angular velocity about the centre is ω.



(a) The linear momentum before the collision is mv and that after the collision is MV.

Thus, $m_2v_0 = m_1V$, or $V = \left(\frac{m_2}{m_1}\right)v_0$

(b) Let A be the centre of the rod when it is at rest. Let AB be the line perpendicular to the plane of the figure. Consider the angular momentum of "the rod plus the particle" system about AB. Initially the rod is at rest. The angular momentum of the particle about AB is

$L = m_2v_0\,(\ell/3)$

After the collision, the particle comes to rest. The angular momentum of the rod about A is

$\vec{L} = \vec{L}_{cm} + m_1\vec{r}_0 \times \vec{V}$

As $\vec{r}_0 \parallel \vec{V}$, $\vec{r}_0 \times \vec{V} = 0$

Thus, $\vec{L} = \vec{L}_{cm}$

Hence the angular momentum of the rod about AB is $L = I\omega = \frac{m_1\ell^2}{12}\omega$. Thus, $\frac{m_2v\ell}{3} = \frac{m_1\ell^2}{12}\omega$

or, $\omega = \frac{4m_2v_0}{m_1\ell}$

9. COMBINED TRANSLATIONAL AND ROTATIONAL MOTION OF A RIGID BODY :

The general motion of a rigid body can be thought of as a sum of two independent motions. A translation of some point of the body plus a rotation about this point. A most convenient choice of the point is the centre of mass of the body as it greatly simplifies the calculations.

☞ Consider a fan inside a train, and an observer A on the platform.

It the fan is switched off while the train moves, the motion of fan is pure translation as each point on the fan undergoes same translation in any time interval.

It fan is switched on while the train is at rest the motion of fan is pure rotation about is axle ; as each point on the axle is at rest, while other points revolve about it with equal angular velocity.

If the fan is switched on while the train is moving, the motion of fan to the observer on the platform is neither pure translation nor pure rotation. This motion is an example of general motion of a rigid body.
Now if there is an observer B inside the train, the motion of fan will appear to him as pure rotation.
Hence we can see that the general motion of fan w.r.t. observer A can be resolved into pure rotation of fan as observed by observer B plus pure translation of observer B (w.r.t. observer A)
Such a resolution of general motion of a rigid body into pure rotation & pure translation is not restricted to just the fan inside the train, but is possible for motion of any rigid system.

9.1 Kinematics of general motion of a rigid body :

For a rigid body as earlier stated value of angular displacement (θ), angular velocity (ω), angular acceleration (α) is same for all points on the rigid body about any other point on the rigid body.
Hence if we know velocity of any one point (say A) on the rigid body and angular velocity of any point on the rigid body about any other point on the rigid body (say ω), velocity of each point on the rigid body can be calculated.

since distance AB is fixed $\vec{V}_{BA} \perp \overrightarrow{AB}$

we know that $\omega = \dfrac{V_{BA\perp}}{r_{BA}}$

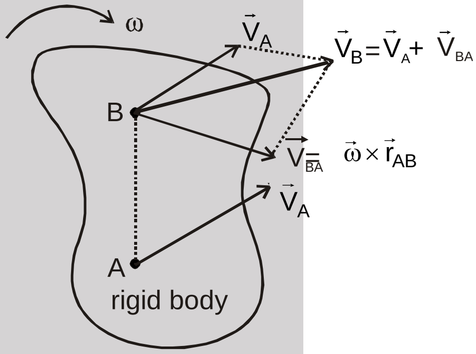
$V_{BA\perp} = V_{BA} = \omega r_{BA}$

in vector form $\vec{V}_{BA} = \vec{\omega} \times \vec{r}_{BA}$

Now from relative velocity : $\vec{V}_{BA} = \vec{V}_B - \vec{V}_A$

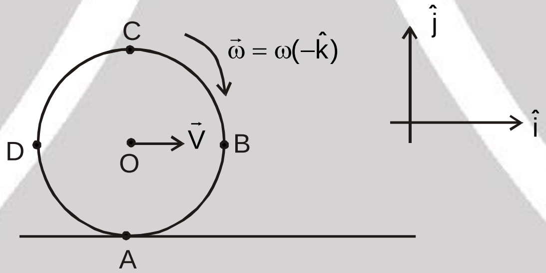
$\vec{V}_B = \vec{V}_A + \vec{V}_{BA} \Rightarrow \vec{V}_B = \vec{V}_A + \vec{\omega} \times \vec{r}_{BA}$

similarly $\vec{a}_B = \vec{a}_A + \vec{\alpha} \times \vec{r}_{BA}$ [for any rigid system]



Solved Examples

Example 51. Consider the general motion of a wheel (radius r) which can be view on pure translation of its center O (with the velocity v) and pure rotation about O (with angular velocity ω)



Find out $\vec{v}_{AO}, \vec{v}_{BO}, \vec{v}_{CO}, \vec{v}_{DO}$ and $\vec{v}_A, \vec{v}_B, \vec{v}_C, \vec{v}_D$

Solution : $\vec{v}_{AO} = \left(\vec{\omega} \times \vec{r}_{AO}\right)$

$\vec{v}_{AO} = \left(\omega(-\hat{k}) \times \vec{OA}\right)$

$\vec{v}_{AO} = \left(\omega(-\hat{k}) \times r(-\hat{j})\right)$

$\vec{v}_{AO} = -\omega r\hat{i}$

similarly $\vec{v}_{BO} = \omega r(-\hat{j})$

$\vec{v}_{CO} = \omega r(\hat{i})$; $\vec{v}_{DO} = \omega r(\hat{j})$

$\vec{v}_A = \vec{v}_O + \vec{v}_{AO} = v\hat{i} - \omega r\hat{i}$

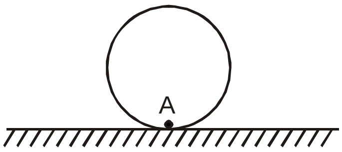
Similarly $\vec{v}_B = \vec{v}_O + \vec{v}_{BO} = v\hat{i} - \omega r\hat{j}$

$\vec{v}_C = \vec{v}_O + \vec{v}_{CO} = v\hat{i} + \omega r\hat{i}$

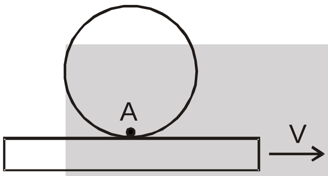
$\vec{v}_D = \vec{v}_O + \vec{v}_{DO} = v\hat{i} + \omega r\hat{j}$

9.2 Pure Rolling (or rolling without sliding) :

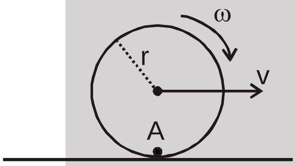
Pure rolling is a special case of general rotation of a rigid body with circular cross section (e.g. wheel, disc, ring, sphere) moving on some surface. Here, there is no relative motion between the rolling body and the surface of contact, at the point of contact

☞ 

Here contact point is A & contact surface is horizontal ground. For pure rolling velocity of A w.r.t. ground = 0 $\Rightarrow V_A = 0$

☞ 

From above figure, for pure rolling, velocity of A w.r.t. to plank is zero $\Rightarrow V_A = V$.

☞ 

From above figure for, pure rolling, velocity of A w.r.t. ground is zero.

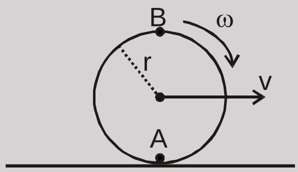
$\Rightarrow$ $v - \omega r = 0$

$v = \omega r$

Similarly $a = \alpha r$

Solved Examples

Example 52. A wheel of radius r rolls (rolling without sleeping) on a level road as shown in figure.



Find out velocity of point A and B

Solution : Contact surface is in rest for pure rolling velocity of point A is zero.

so $v = \omega\, r$

velocity of point B = $v + \omega r = 2v$

9.3 Dynamics of general motion of a rigid body :

This motion can be viewed as translation of centre of mass and rotation about an axis passing through centre of mass

If I_{CM} = Moment of inertia about this axis passing through COM

τ_{cm} = Net torque about this axis passing through COM

$\vec{a}_{CM}$ = Acceleration of COM

$\vec{v}_{CM}$ = Velocity of COM

$\vec{F}_{ext}$ = Net external force acting on the system.

$\vec{P}_{system}$ = Linear momentum of system.

$\vec{L}_{CM}$ = Angular momentum about centre of mass.

$\vec{r}_{CM}$ = Position vector of COM w.r.t. point A.

then (i) $\vec{\tau}_{cm} = I_{cm}\ \vec{\alpha}$

(ii) $\vec{F}_{ext} = M\vec{a}_{cm}$

(iii) $\vec{P}_{system} = M\vec{v}_{cm}$

(vi) Total K.E.= $\frac{1}{2}Mv_{cm^2} + \frac{1}{2}I_{cm}\omega^2$

(v) $\vec{L}_{CM} = I_{CM}\vec{\omega}$

(vi) Angular momentum about point A = $\vec{L}$ about C.M. + $\vec{L}$ of C.M. about A

$\vec{L}_A = I_{cm}\ \vec{\omega} + \vec{r}_{cm} \times M\vec{v}_{cm}$

☞ $\frac{dL_A}{dt} = \frac{d}{dt}(I_{cm}\vec{\omega} + \vec{r}_{cm} \times M\vec{v}_{cm}) \neq I_A\frac{d\vec{\omega}}{dt}$. Notice that torque equation can be applied to a rigid body in a general motion only and only about an axis through centre of mass.

Solved Examples

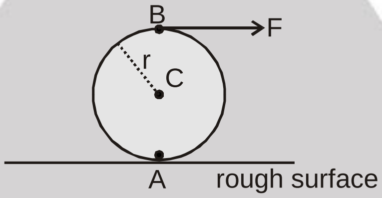
Example 53. A uniform sphere of mass 200 g rolls without slipping on a plane surface so that its centre moves at a speed of 2.00 cm/s. Find its kinetic energy.

Solution : As the sphere rolls without slipping on the plane surface, its angular speed about the centre is $\omega = \frac{v_{cm}}{r}$. The kinetic energy is

$K = \frac{1}{2}\ I_{cm}\omega^2 + \frac{1}{2}\ Mv_{cm}^2 \quad = \frac{1}{2}.\frac{2}{5}Mr^2\omega^2 + \frac{1}{2}Mv_{cm}^2$

$= \frac{1}{5}\ Mv_{cm}^2 + \frac{1}{2}Mv_{cm}^2 = \frac{7}{10}Mv_{cm}^2 = \frac{7}{10}$ (0.200 kg) (0.02 m/s)^2 = 5.6 × 10^{-5} J.

Example 54. A constant force F acts tangentially at the highest point of a uniform disc of mass m kept on a rough horizontal surface as shown in figure. If the disc rolls without slipping, calculate the acceleration of the centre (C) and point A and B of the disc.



Solution : The situation is shown in figure. As the force F rotates the disc, the point of contact has a tendency to slip towards left so that the static friction on the disc will act towards right. Let r be the radius of the disc and a be the linear acceleration of the centre of the disc. The angular acceleration about the centre of the disc is α = a/r, as there is no slipping.

For the linear motion of the centre,

F + f = ma ......(i)

and for the rotational motion about the centre,

$Fr - f\,r = I\,\alpha = \left(\frac{1}{2}mr^2\right)\left(\frac{a}{r}\right)$ or, $F - f = \frac{1}{2}ma$ ......(ii)

From (i) and (ii),

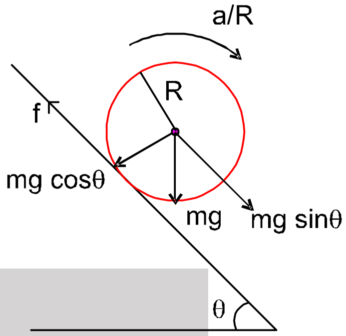
$2F = \frac{3}{2}ma$ or $a = \frac{4F}{3m}$

Acceleration of point A is zero.

Acceleration of point B is $2a = 2\left(\frac{4F}{3m}\right) = \left(\frac{8F}{3m}\right)$ **Ans.**

Example 55. A circular rigid body of mass m, radius R and radius of gyration (k) rolls without slipping on an inclined plane of a inclination θ. Find the linear acceleration of the rigid body and force of friction on it. What must be the minimum value of coefficient of friction so that rigid body may roll without sliding?

Solution : If a is the acceleration of the centre of mass of the rigid body and f the force of friction between sphere and the plane, the equation of translatory and rotatory motion of the rigid body will be.



$mg \sin\theta - f = ma$ (Translatory motion)

$fR = I\alpha$ (Rotatory motion)

$$f = \frac{I\alpha}{R}$$

$I = mk^2$, due to pure rolling $a = \alpha R$

$$mg \sin\theta - \frac{I\alpha}{R} = m\alpha R$$

$$mg \sin\theta = m\alpha R + \frac{I\alpha}{R}$$

$$mg \sin\theta = m\alpha R + \frac{mk^2\alpha}{R}$$

$$mg \sin\theta = ma + \frac{mk^2\alpha}{R}$$

$$mg \sin\theta = a\left[\frac{R^2+k^2}{R^2}\right]$$

$$a = \frac{g\sin\theta}{\left[\frac{R^2+k^2}{R^2}\right]} \quad ; \quad a = \frac{g\sin\theta}{\left(1+\frac{k^2}{R^2}\right)}$$

$$f = \frac{I\alpha}{R}$$

$$f = \frac{mk^2a}{R^2} \Rightarrow \frac{mg\,k^2\sin\theta}{R^2+k^2}$$

$f \le \mu N$

$$\frac{mk^2}{R^2}a \le \mu \le mg\cos\theta$$

$$R^2\frac{k^2}{R^2} \times \frac{g\sin\theta}{(k^2+R^2)} \le \mu g\cos\theta$$

$$\mu \ge \frac{\tan\theta}{\left[1+\frac{R^2}{k^2}\right]} \Rightarrow \mu_{min} = \frac{\tan\theta}{\left[1+\frac{R^2}{k^2}\right]}$$

Note : From above example if rigid bodies are solid cylinder, hollow cylinder, solid sphere and hollow sphere.

(1) Increasing order of acceleration.

$a_{\text{solid sphere}} > a_{\text{hollow sphere}} > a_{\text{solid cylinder}} > a_{\text{hollow cylinder}}$

(2) Increasing order of required friction force for pure rolling.

$f_{\text{hollow cylinder}} > f_{\text{hollow sphere}} > f_{\text{solid cylinder}} > f_{\text{solid sphere}}$

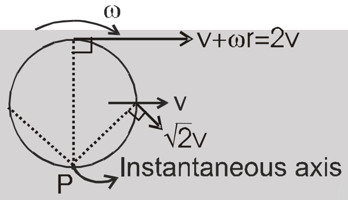
(3) Increasing order of required minimum friction coefficient for pure rolling.

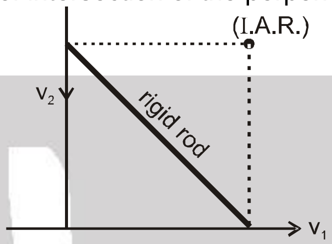
$\mu_{\text{hollow cylinder}} > \mu_{\text{hollow sphere}} > \mu_{\text{solid cylinder}} > \mu_{\text{solid sphere}}$

9.4 Instantaneous axis of rotation :

It is the axis about which the combined translational and rotational motion appears as pure rotational motion.
The combined effect of translation of centre of mass and rotation about an axis through the centre of mass is equivalent to a pure rotation with the same angular speed about a stationary axis ; this axis is called instantaneous axis of rotation. It is defined for an instant and its position changes with time.
eg. In pure rolling the point of contact with the surface is the instantaneous axis of rotation.

Geometrical construction of instantaneous axis of rotation (I.A.R). Draw velocity vector at any two points on the rigid body. The I.A.R. is the point of intersection of the perpendicular drawn on them.





☞ In case of pure rolling the lower point is instantaneously axis of rotation.
The motion of body in pure rolling can therefore by analysed as pure rotation about this axis. Consequently

$\tau_P = I_P\alpha$

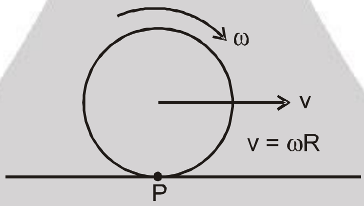
$\alpha_P = I_P\omega$

K.E. = $1/2\ I_P\ \omega^2$

Where I_P is moment of inertial instantaneous axis of rotation passing through P.

Solved Examples

Example 56.



Prove that kinetic energy = $1/2\ I_P\ \omega^2$

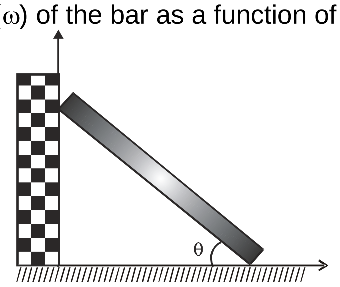
Solution : K.E. = $\frac{1}{2} I_{cm}\ \omega^2 + \frac{1}{2} Mv_{cm}^2 = \frac{1}{2} I_{cm}\ \omega^2 + \frac{1}{2} M\omega^2R^2$

$\frac{1}{2}(I_{cm} + MR^2)\ \omega^2$

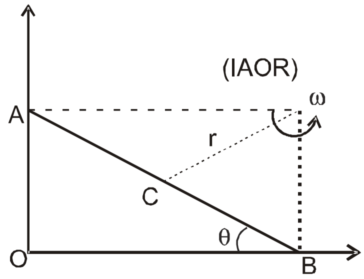
$\frac{1}{2}\left(I_{contact\ \ point}\right)\ \omega^2$

☞ Notice that pure rolling of uniform object equation of torque can also be applied about the contact point.

Example 57. A uniform bar of length ℓ and mass m stands vertically touching a vertical wall (y-axis). When slightly displaced, its lower end begins to slide along the floor (x-axis). Obtain an expression for the angular velocity (ω) of the bar as a function of θ. Neglect friction everywhere.



Solution : The position of instantaneous axis of rotation (IAOR) is shown in figure.



$$C = \left(\frac{\ell}{2}\cos\theta,\ \ \frac{\ell}{2}\sin\theta\right)$$

$r = \frac{\ell}{2}$ = half of the diagonal

All surfaces are smooth. Therefore, mechanical energy will remain conserved.

$\therefore$ Decrease in gravitational potential energy of bar = increase in rotational kinetic energy of bar about IAOR.

$\therefore$ $mg\ \frac{\ell}{2}(1 - \sin\theta) = \frac{1}{2}\ I\omega^2$ ....(1)

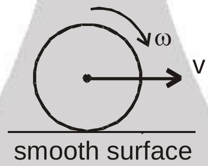
Here, $I = \frac{m\ell^2}{12} + mr^2$ (about IAOR) or $I = \frac{m\ell^2}{12} + \frac{m\ell^2}{4} = \frac{m\ell^2}{3}$

Substituting in Eq. (1), we have

$mg\ \frac{\ell}{2}\ (1 - \sin\theta) = \frac{1}{2}\left(\frac{m\ell^2}{3}\right)\omega^2$ or $\omega = \sqrt{\frac{3g\ \ (1-\sin\theta)}{\ell}}$ **Ans.**

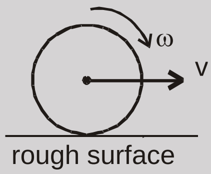
The nature of friction in the following cases assume body is perfectly rigid

(i) $v = \omega R$



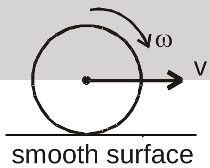
No friction and pure rolling.

(ii) $v = \omega R$



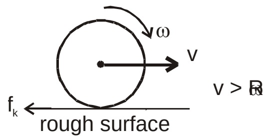
No friction and pure rolling (If the body is not perfectly rigid, then there is a small friction acting in this case which is called rolling friction)

(iii) $v > \omega R$ or $v < \omega R$



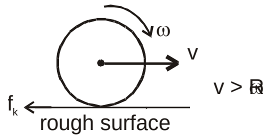
No friction force but not pure rolling.

(iv) $v > \omega R$



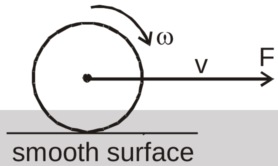
There is Relative Motion at point of contact so Kinetic Friction, $f_k = \mu N$ will act in backward direction. This kinetic friction decrease v and increase ω, so after some time $v = \omega R$ and pure rolling will resume like in case (ii).

(v) $v < \omega R$



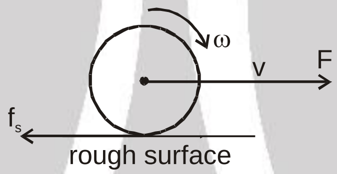
There is Relative Motion at point of contact so Kinetic Friction, $f_k = \mu N$ will act in forward direction. This kinetic friction increase v and decrease ω, so after some time $v = \omega R$ and pure rolling will resume like in case (ii).

(vi) $v = \omega R$ (initial)



No friction and no pure rolling.

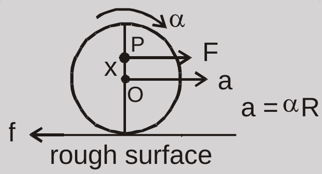
(vii) $v = \omega R$ (initial)



Static friction whose value can be lie between zero and $\mu_s N$ will act in backward direction. If coefficient of friction is sufficiently high, then f_s compensates for increasing v due to F by increasing ω and body may continue in pure rolling with increases v as well as ω.

Solved Examples

Example 58. A rigid body of mass m and radius r rolls without slipping on a rough surface. A force is acting on a rigid body x distance from the centre as shown in figure. Find the value of x so that static friction is zero.



Solution : Torque about centre of mass

$Fx = I_{cm}\alpha$ .....(1)

$F = ma$ .....(2)

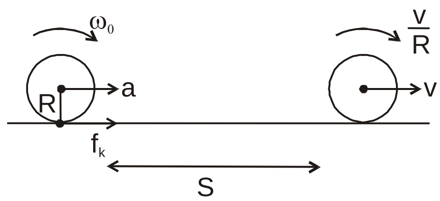
From eqn. (1) & (2)

$max = I_{cm}\,\alpha \qquad (a = \alpha R)\ ; \quad x = \dfrac{I_{cm}}{mR}$

Note : For pure rolling if any friction is required then friction force will be statics friction. It may be zero, backward direction or forward direction depending on value of x. If F below the point P then friction force will act in backward direction or above the point P friction force will act in forward direction.

Example 59. A cylinder is given angular velocity ω_0 and kept on a horizontal rough surface the initial velocity is zero. Find out distance travelled by the cylinder before it performs pure rolling and work done by friction force

Solution : $\mu Mg\,R = \dfrac{MR^2\alpha}{2}$



$\alpha = \dfrac{2\mu g}{R}$ .....(1)

Initial velocity u = 0

$v_2 = u_2 + 2as$

$v^2 = 2as$ .....(2)

$f_K = Ma$

$\mu Mg = Ma$

$a = \mu g$ .....(3)

$\omega = \omega_0 - \alpha t$

from equation (1) $\omega = \omega_0 - \frac{2\mu g}{R} t$

$v = u + at$

from equation (3) $v = \mu g t$

$\omega = \omega_0 - \frac{2v}{R}$

$\omega = \omega_0 - 2\omega$

$\omega = \frac{\omega_0}{3}$

from equation (2)

$$\left(\frac{\omega_0 R}{3}\right)^2 = (2as) = 2\mu gs$$

$$s = \left(\frac{\omega_0^2 \; R^2}{18 \; \mu g}\right)$$

work done by the friction force $w = (-f_k R\, d\theta + f_k \Delta s)$

$$-\mu mg R \Delta\theta + \frac{\mu mg \times \omega_0^2 R^2}{18 \; \mu g}$$

$$\Delta\theta = \omega_0 \times t - \frac{1}{2}\alpha t^2 = \omega_0 \times \left(\frac{\omega_0 R}{3\mu g}\right) - \frac{1}{2} \times \frac{2\mu g}{R}\left(\frac{\omega_0 R}{3\mu g}\right)^2$$

$$\frac{\omega_0^2 R}{3\mu g} - \frac{\omega_0^2 R}{9\mu g}$$

$$\frac{2\omega_0^2 R}{9\mu g}$$

$$-\mu mg \times R \frac{2\omega_0^2 R}{9\mu g} + \mu mg \times \frac{\omega_0^2 R^2}{18\mu g}$$

$$-\frac{2m\omega_0^2 R^2}{9} + \frac{m\omega_0^2 R^2}{18}$$

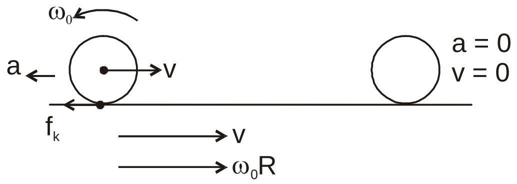
$$\frac{-3m\omega_0^2 R^2}{18} = -\frac{m\omega_0^2 R^2}{6}$$

Alternative Solution

Using work energy theorm $w_g + w_a + w_{f_k} = \Delta K$

$$w_{f_k} = \left[\frac{1}{2}m\left(\frac{\omega_0 R}{3}\right)^2 + \frac{1}{2}\frac{mR^2}{2}\times\left(\frac{\omega_0}{3}\right)^2\right] - \left[\frac{1}{2}\frac{mR^2}{2}\times\omega_0^2\right] = \left(-\frac{m\omega_0{}^2 R^2}{6}\right)$$

Example 60. A hollow sphere is projected horizontally along a rough surface with speed v and angular velocity ω_0 find out the ratio. So that the sphere stope moving after some time.



Solution : Torque about lowest point of sphere.

$f_k \times R = I\alpha$

$\mu mg \times R = \frac{2}{3} mR^2 \alpha$

$\alpha = \frac{3\mu g}{2R}$ angular acceleration in opposition direction of angular velocity.

$\omega = \omega_0 - \alpha t$ (final angular velocity $\omega = 0$)

$\omega_0 = \frac{3\mu g}{2R} \times t$

$t = \frac{\omega_0 \times 2R}{3\mu g}$

acceleration 'a = μg'

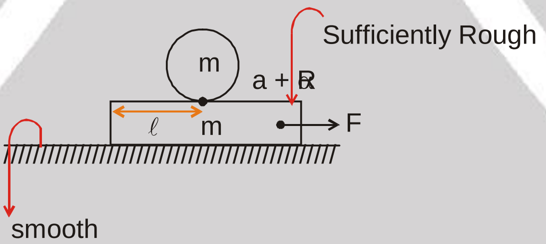
$v_f = v - at$ (final velocity $v_f = 0$)

$v = \mu g \times t$; $t = \frac{v}{\mu g}$

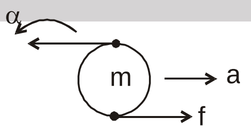
To stop the sphere time at which v & ω are zero, should be same.

$$\frac{v}{\mu g} = \frac{2\omega_0 R}{3\mu g} = \frac{v}{\omega_0} = \frac{2R}{3}$$

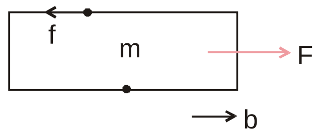
9.5 Rolling on moving surface



Friction on the plate backward or on cylinder friction forward so cylinder move forward.



Because of pure rolling static friction f.



$fR = \frac{mR^2}{2} \alpha$

$\alpha = \frac{2f}{mR}$

$f = ma$

$F - f = mb$

$F = m(a + b)$

$a = \frac{\alpha R}{2}$

At contact point

$b = a + \alpha R$

$b = \frac{3\alpha R}{2}$

$b = 3a$

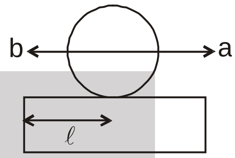
$F = 4ma$

$a = \frac{F}{4m}$

$b = \frac{3F}{4m}$

w.r.t. plate distance is covered = ℓ

and acceleration w.r.t. plate (b – a)



$\ell = \frac{1}{2}(b - a)\, t^2$

$\ell = \frac{1}{2} \times 2at^2 = t = \sqrt{\frac{a \times \ell}{F}} = 2\sqrt{\frac{m\ell}{F}}$

Solved Examples

Example 61. A solid sphere is released from rest from the top of an incline of inclination θ and length. If the sphere rolls without slipping, what will be its speed when it reaches the bottom ?

Solution : Let the mass of the sphere be m and its radius r. Suppose the linear speed of the sphere when it reaches the bottom is v. As the sphere rolls without slipping, its angular speed about its axis is $\omega = v/r$. The kinetic energy at the bottom will be

$$K = \frac{1}{2}I\omega^2 + \frac{1}{2}mv^2 = \frac{1}{2}\left(\frac{2}{5}mr^2\right)\omega^2 + \frac{1}{2}mv^2 = \frac{1}{5}mv^2 + \frac{1}{2}mv^2 = \frac{7}{10}mv^2$$

This should be equal to the loss of potential energy mg sinθ. Thus,

$$\frac{7}{10}mv^2 = mg\ell \sin\theta \quad \text{or} \quad v = \sqrt{\frac{10}{7}g\ell\sin\theta}$$

Example 62. There are two cylinders of radii R_1 and R_2 having moments of inertia I_1 and I_2 about their respective axes as shown in figure. Initially, the cylinders rotate about their axes with angular speed ω_1 and ω_2 as shown in the figure. The cylinders are moved closed to touch each other keeping the axes parallel. The cylinders first slip over each other at the contact but the slipping finally ceases due to the friction between them. Calculate the angular speeds of the cylinders after the slipping ceases.

Solution : When slipping ceases, the linear speeds of the points of contact of the two cylinders will be equal. If ω'_1 and ω'_2 be the respective angular speeds, we have

$\omega'_1 R_1$ and $\omega'_2 R_2$ ....(i)

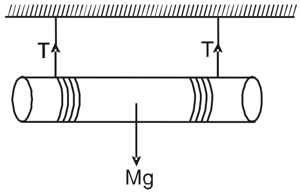
The change in the angular speed is brought about by the frictional force which acts as long as the slipping exists. If this force f acts for a time t, the torque on the first cylinder is fR_1 and that on the second is fR_2. Assuming $\omega_1 > \omega_2$ the corresponding angular impulses are $-fR_1t$ and fR_2t, We, there fore, have

$- f R_1 t = I_1 (\omega'_1 - \omega_1)$ and $fR_2 t = I_2 (\omega'_2 - \omega_2)$

or, $-\frac{I_1}{R_1}(\omega'_1 - \omega_1) = \frac{I_2}{R_2}(\omega'_2 - \omega_2)$ ....(ii)

Solving (i) and (ii) $\omega'_1 = \frac{I_1\ \omega_1\ R_2 + I_2\ \omega_2\ R_1}{I_2\ R_1^2\ + I_1\ R_2^2} R_2$ and $\omega'_2 = \frac{I_1\ \omega_1\ R_2 + I_2\ \omega_2\ R_1}{I_2\ R_1^2\ + I_1\ R_2^2} R_1$

Example 63. A hollow cylinder of mass m is suspended through two light strings rapped around it as shown in figure. Calculate (a) the tension T in the string and (b) the speed of the cylinder as it falls through a distance ℓ.



Solution : The portion of the strings between the ceiling and the cylinder is at rest. Hence the points of the cylinder where the strings leave it are at rest. The cylinder is thus rolling without slipping on the strings. Suppose the centre of the cylinder falls with an acceleration a. The angular acceleration of the cylinder about its axis is α = a/R, as the cylinder does not slip over the strings.

The equation of motion for the centre of mass of the cylinder is

$$Mg - 2T = Ma \qquad ....(i)$$

and for the motion about the centre of mass, it is

$$2\,Tr = (Mr^2\alpha) = Mra \quad \text{or} \quad 2T = Ma.$$

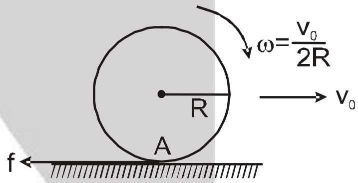
From (i) and (ii),

$$a = \frac{g}{2} \text{ and } T = \frac{Mg}{4}$$

As the centre of the cylinder starts moving from rest, the velocity after it has fallen through a distance ℓ is given by

$$v^2 = 2\left(\frac{g}{2}\right)\ell \qquad \text{or} \qquad v = \sqrt{g\ell}$$

Example 64. A hollow sphere of mass M and radius R as shown in figure slips on a rough horizontal plane. At some instant it has linear velocity v_0 and angular velocity about the centre $\frac{v_0}{2R}$ as shown in figure. Calculate the linear velocity after the sphere starts pure rolling.



Solution : Velocity of the centre = v_0 and the angular velocity about the centre = $\frac{v_0}{2R}$. Thus $v_0 > \omega_0 R$. The sphere slips forward and thus the friction by the plane on the sphere will act backward. As the friction is kinetic, its value is μN = μMg and the sphere will be decelerated by a_{cm} = f/M. Hence,

$$v(t) = v_0 - \frac{f}{M}t \qquad ....(i)$$

This friction will also have a torque I' = fr about the centre. This torque is clockwise and in the direction of ω_0. Hence the angular acceleration about the centre will be

$$\alpha = f\,\frac{R}{(2/3)MR^2} = \frac{3f}{2MR}$$

and the clockwise angular velocity at time t will be $\omega(t) = \omega_0 + \frac{3f}{2MR}t = \frac{v_0}{2R} + \frac{3f}{2MR}t.$

Pure rolling starts when $v(t) = R\omega(t)$ i.e., $v(t) = \frac{v_0}{2} + \frac{3f}{2M}t.$ ....(ii)

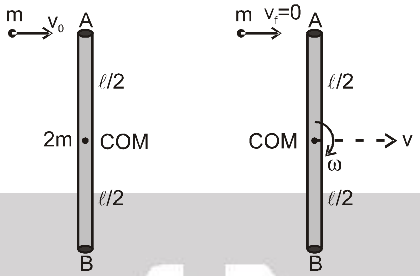
Eliminating t from (i) and (ii), $\frac{3}{2}v(t) + v(t) = \frac{3}{2}v_0 + \frac{v_0}{2}$ or $v(t) = \frac{2}{5} \times 2v_0 = \frac{4}{5}v_0.$

Thus, the sphere rolls with linear velocity $4v_0/5$ in the forward direction.

Example 65. A rod AB of mass 2m and length ℓ is lying on a horizontal frictinless surface. A particle of mass m traveling along the surface hits the end 'A' of the rod with a velocity v_0 in a direction perpendicular to AB. The collisin is elastic. After the collision the particle comes to rest. Find out after collision

(a) Velocity of centre of mass of rod (b) Angular velocity.

Solution : (a) Let just after collision lthe sped of COM of rod is v and angular velocity about COM is ω.



Before collision After collision

External force on the system (rod + mass) in horizontal plane is zero

Apply conservation of linear momentum in x direction

$mv_0 = 2mv$ ....(1)

Net torque on the system about any point is zero

Apply conservation of angular momentum about COM of rod.

$$mv_0\frac{\ell}{2} = I\omega \quad \Rightarrow \quad mv_0\frac{\ell}{2} = \frac{2m\ell^2}{12}\omega$$

$$mv_0 = m\omega\frac{\ell}{3} \qquad ....(2)$$

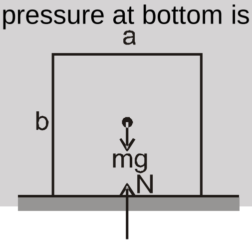
From eq (1) velocity of centre of mass $v = \frac{v_0}{2}$

From eq (2) anuglar velocity $\omega = \frac{3v_0}{\ell}$

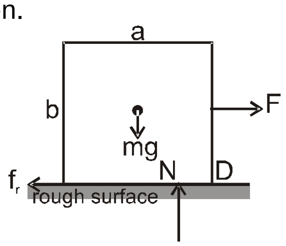
10. TOPPLING :

In many situations an external force is applied to a body to cause it to slide along a surface. In certain cases, the body may tip over before sliding ensues. This is known as topping.

(1) There is a no horizontal force so pressure at bottom is uniform and normal is colinear with mg.



(2) If a force is applied at COM, pressure is not uniform Normal shifts right so that torque of N can counter balance torque of friction.



$F_{max} = f_r$

N = mg

$f_r \cdot b/2 = N \cdot a/2 \Rightarrow f_r = Na/b = mg\, a/b$, $F_{max} = mg\, a/b$

(3) If surface is not sufficiently rough and the body slides before F is increased to F_{max} = mg a/b then body will slide before toppling. Once body starts sliding friciton becomes constant and hence no topping. This is the case if

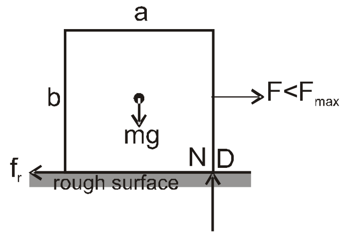
$F_{max} > f_{limit}$

$\Rightarrow$ mg a/b > μmg

μ < a/b

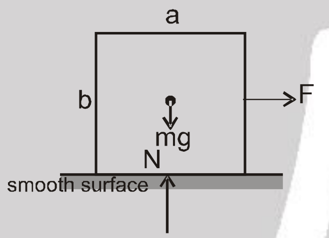
Condition for toppling when $\mu \geq$ a/b in this case body will topple if F > mg a/b

but if μ < a/b, body will not topple any value of F applied a COM



Solved Examples

Example 66.



Find out minimum value of F for toppling

Solution : Never topple

Example 67. A uniform cube of side 'a' and mass m rests on a rough horizontal table. A horizontal force F is applied normal to one of the faces at a point directly below the centre of the face, at a height $\frac{a}{4}$ above the base.

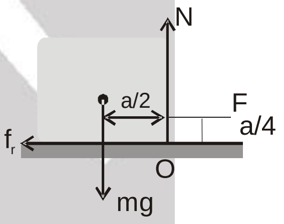
(i) What is the minimum value of F for which the cube begins to tip about an edge?

(ii) What is the minimum value of μ_s so that toppling occures.

(iii) If $\mu = \mu_{min}$, find minimum force for topping.

(iv) Minimum μ_s so that F_{min} can cause toppling.

Solution : (i) In the limiting case normal reaction will pass through O. The cube will tip about O if torque of Fabout O exceeds the torque of mg.



Hence, F $\left(\frac{a}{4}\right) >$ mg $\left(\frac{a}{2}\right)$ or F > 2 mg

therefore, minimum value of F is 2 mg

(ii) In this case since it is not acting at COM, toppling can occur even after body started slinding because increasing the the torque of F about COM.hence μ_{min} = 0,

(iii) Now body is sliding before toppling, O is not I.A.R., torque equation can not be applied across it. It can now be applied about COM.

$$F \times \frac{a}{4} = N \times \frac{a}{2} \qquad .....(1)$$

$$N = mg \qquad .....(2)$$

from (1) and (2)

F = 2 mg

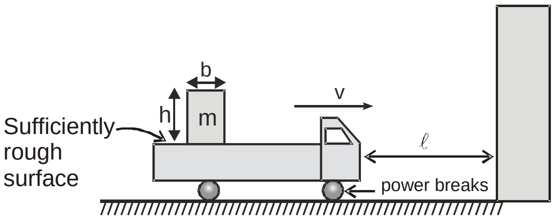
(iv) F > 2 mg .... (1) (from sol. (i))

N = mg ....(2)

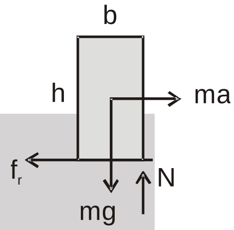
F = μ_sN = $\mu_s$mg .... (3) from (1) and (2)

μ_s = 2

Example 68. Find minimum value of ℓ so that truck can avoid the dead end, without toppling the block kept on it.



Solution : $ma\frac{h}{2} \leq mg\frac{b}{2} \Rightarrow a \leq \frac{b}{h}g$



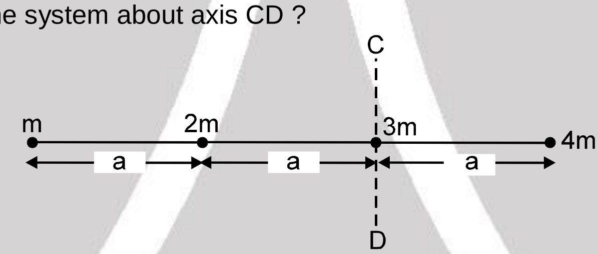
Final velocity of truck is zero. So that

$0 = v^2 - 2(\frac{b}{h}g)\,\ell$

$\ell = \frac{h}{2b}\frac{v^2}{g}$

Self Practice Problems

Problem 1. Four point masses are connected by a massless rod as shown in figure. Find out the moment of inertia of the system about axis CD ?



Solution : $I_1 = m(2a)^2$

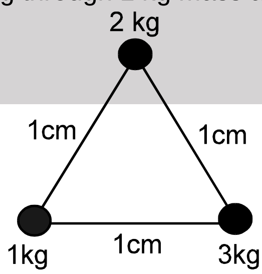
$I_2 = 2ma^2$

$I_3 = 0$

$I_4 = 4ma^2$

$I_{CD} = I_1 + I_2 + I_3 + I_4 = 10\ ma^2$ **Ans.**

Problem 2. Three point masses are located at the corners of an equilibrium triangle of side 1 cm. Masses are of 1, 2 & 3kg respectively and kept as shown in figure. Calculate the moment of Inertia of system about an axis passing through 1 kg mass and perpendicular to the plane of triangle?



Solution : Moment of inertia of 2 kg mass about an axis passing through 1 kg mass

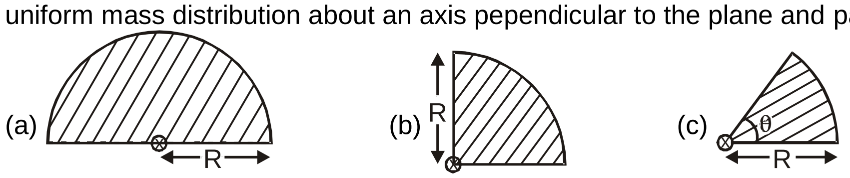
$I_1 = 2 \times (1\times10^{-2})^2 = 2\times10^{-4}$

Moment of inertia of 3 kg mass about an axis passing through 1 kg mass

$I_2 = 3 \times (1\times10^{-2})^2 = 3\times10^{-4}$

$I = I_1 + I_2 = 5 \times 10^{-4}$ kgm^2

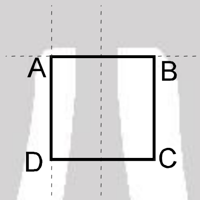
Problem 3. Calculate the moment of Inertia of figure shown each having mass M, radius R and having uniform mass distribution about an axis pependicular to the plane and passing through centre?



Solution : $dI = dm\,\frac{R^2}{2}$

$I = \int dI = \frac{R^2}{2}\int dm = \frac{MR^2}{2}$

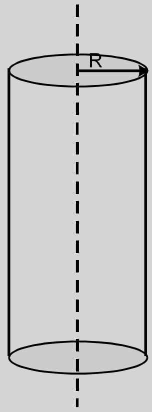
Problem 4. Find the moment of inertia of the uniform square plate of side 'a' and mass M about the axis AB.



Solution : $dI = dm\,\frac{a^2}{3}$

$I = \int dI = \frac{a^2}{3}\int dm = \frac{Ma^2}{3}$

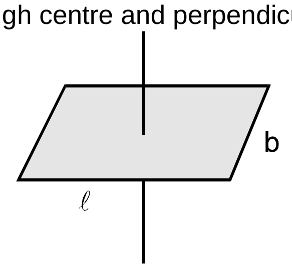
Problem 5. Calculate the moment of inertia of a uniform solid cylinder of mass M, radius R and length ℓ about its axis.



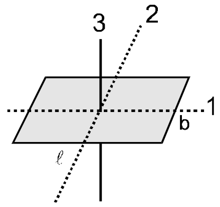
Solution : Each segment of cylinder is solid disc so $\int dI = \int dm\frac{R^2}{2}$

$I = \frac{MR^2}{2}$ **Ans.**

Problem 6. Find the moment of inertia of a uniform rectangular plate of mass M, edges of length 'ℓ' and 'b' about its axis passing through centre and perpendicular to it.



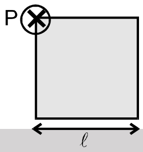
Solution : Using perpendicular axis theorem $I_3 = I_1 + I_2$



$$I_1 = \frac{Mb^2}{12}$$

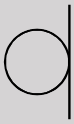
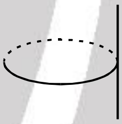
$$I_2 = \frac{M\ell^2}{12} \quad ; \quad I_3 = \frac{M(\ell^2 + b^2)}{12}$$

Problem 7. Find the moment of inertia of a uniform square plate of mass M, edge of length 'ℓ' about its axis passing through P and perpendicular to it.



Solution : $I_P = \frac{M\ell^2}{6} + \frac{M\ell^2}{2} = \frac{2M\ell^2}{3}$

Problem 8. Find out the moment of inertia of a ring having uniform mass distribution of mass M & radius R about an axis which is tangent to the ring and (i) in the plane of the ring (ii) perpendicular to the plane of the ring.

(i)  (ii) 

Solution : **(i)** Moment of inertia about an axis passing through centre of ring and plane of the ring $I_1 = \frac{MR^2}{2}$

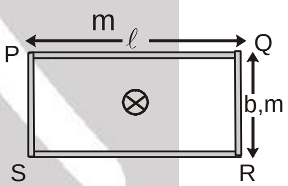
Using parallel axis theorem $I' = I_1 + MR^2 = \frac{3MR^2}{2}$

(ii) Moment of inertia about an axis passing through centre of ring and perpendicular to palne of the ring

$I_C = MR^2$

Using parallel axis theorem $I'' = I_C + MR^2 = 2MR^2$

Problem 9. Calculate the moment of inertia of a rectangular frame formed by uniform rods having mass m each as shown in figure about an axis passing through its centre and perpendicular to the plane of frame ? Also find moment of inertia about an axis passing through PQ ?



Solution : (i) Moment of inertia about an axis passing through its centre and perpendicular to the plane of frame

$I_C = I_1 + I_2 + I_3 + I_4$

$I_1 = I_3$, $I_2 = I_4$

$I_C = 2I_1 + 2I_2$

$$I_1 = \frac{m\ell^2}{12} + m\left(\frac{b}{2}\right)^2 \Rightarrow I_2 = \frac{mb^2}{12} + m\left(\frac{\ell}{2}\right)^2 \quad \text{so, } I_C = \frac{2m}{3}(\ell^2 + b^2)$$

(ii) M.I. about axis PQ of rod PQ $I_1 = 0$

M.I. about axis PQ of rod PS $I_2 = \frac{mb^2}{2}$

M.I. about axis PQ of rod QR $I_3 = \frac{mb^2}{2}$

M.I. about axis PQ of rod SR $I_4 = mb^2$

$$I = I_1 + I_2 + I_3 + I_4 = \frac{5mb^2}{3}$$

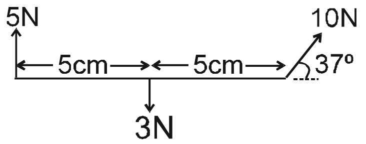
Problem 10. In the previous question, during the motion of particle from P to Q. Torque of gravitational force about P is :

(A) increasing (B) decreasing

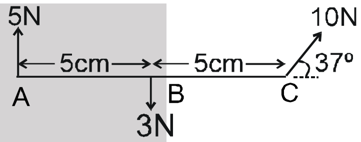
(C) remains constant (D) first increasing then decreasing

Solution : Increasing because distance from point P is increasing.

Problem 11. Determine the point of application of force, when forces are acting on the rod as shown in figure.



Solution : Torque of B about A τ_1 = 3N × 5 = 15N cm (clockwise)



Torque of C about A τ_2 = 6N × 10 = 60 N cm (anticlockwise)

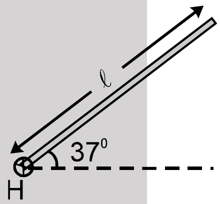
Resultant force perpendicular to the rod F = 8 N

$\tau_1 + \tau_2 = F x$ (x = distance from point A)

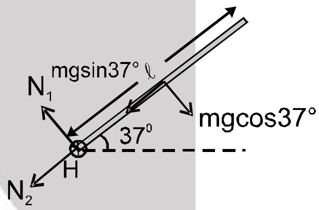
$-15 + 60 = 8x$

$x = 45/8 = 5.625$ cm

Problem 12. A uniform rod of mass m and length ℓ can rotate in vertical plane about a smooth horizontal axis hinged at point H. Find angular acceleration α of the rod just after it is released from initial position making an angle of 37° with horizontal from rest? Find force exerted by the hinge just after the rod is released from rest.



Solution : Torque about hing = $\tau_H = I\,\alpha$



$$mg\cos 37\ \frac{\ell}{2} = \frac{m\ell^2}{3}\alpha$$

$\alpha = 6g / 5\ell$

$$a_t = \alpha\frac{\ell}{2} = \frac{3g}{5}$$

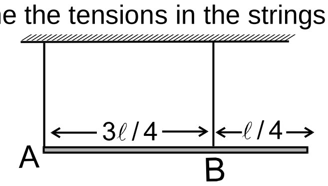
$mg\cos 37 - N_1 = ma_t$

$$N_1 = \frac{mg}{5}$$

angular velocity of rod is zero. so $N_2 = mg\sin 37° = 3mg/5$

$$N = \sqrt{N_1^2 + N_2^2} = \sqrt{\left(\frac{mg}{5}\right)^2 + \left(\frac{3mg}{5}\right)^2} = \frac{mg\sqrt{10}}{5}$$

Problem 13. A uniform rod of length, mass m is hung from two strings of equal length from a ceiling as shown in figure. Determine the tensions in the strings ?



Solution : $T_A + T_B = mg$ ....(i)

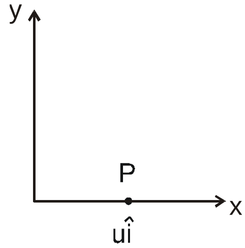
Torque about point A is zero

$$\text{So, } T_B \times \frac{3\ell}{4} = mg\frac{\ell}{2} \quad ....(ii)$$

From eq. (i) & (ii),

$T_A = mg/3,\ T_B = 2mg/3$

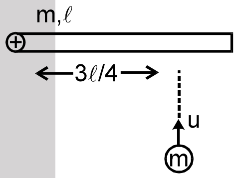
Problem 14. A particle of mass m starts moving from origin with a constant velocity $u\hat{i}$ find out its angular momentum about origin at this moment. What will be the answer later on? What will be the answer if the speed increases.



Solution : $\vec{L} = \vec{r} \times \vec{p}$

$\vec{L} = r\hat{i} \times mu\hat{i} = 0$

Problem 15. A uniform rod of mass m and length ℓ can rotate freely on a smooth horizontal plane about a vertical axis hinged at point H. A point mass having same mass m coming with an initial speed u perpendicular to the rod, strikes the rod and sticks to it at a distance of $3\ell/4$ from hinge point. Find out the angular velocity of the rod just after collision ?

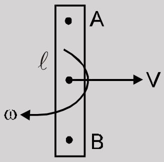


Solution : Angular Momentum about hinge

$L_i = L_f$

$$mu\left(\frac{3\ell}{4}\right) = \left(\frac{m\ell^2}{3} + m\left(\frac{3\ell}{4}\right)^2\right)\omega \qquad \omega = \frac{36u}{43\ell}$$

Problem 16. Uniform & smooth Rod of length ℓ is moving with a velocity of centre v and angular velocity ω on smooth horizontal surface. Findout velocity of point A and B.



Solution : velocity of point A w.r.t. center is $\omega\frac{\ell}{2}$

velocity of point A w.r.t. ground $V_A = V + \omega\frac{\ell}{2}$

velocity of point B w.r.t. center is $-\omega\frac{\ell}{2}$

velocity of point B w.r.t. ground $V_A = V - \omega\frac{\ell}{2}$