QUADRATIC EQUATION

A man is like a fraction whose numerator is what he is and whose denominator is what he thinks of himself. The larger the denominator the smaller the fraction.........
Tolstoy, Count Lev Nikolgevich

1. Polynomial :

A function f defined by $f(x) = a_n x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$
where $a_0, a_1, a_2, ..., a_n \in R$ is called a polynomial of degree n with real coefficients ($a_n \neq 0$, $n \in W$). If $a_0, a_1, a_2, ..., a_n \in C$, it is called a polynomial with complex coefficients.

2. Quadratic polynomial & Quadratic equation :

A polynomial of degree 2 is known as quadratic polynomial. Any equation f(x) = 0, where f is a quadratic polynomial, is called a quadratic equation. The general form of a quadratic equation is

$ax^2 + bx + c = 0$ ....(i)

Where a, b, c are real numbers, $a \neq 0$.
If a = 0, then equation (i) becomes linear equation.

3. Difference between equation & identity :

If a statement is true for all the values of the variable, such statements are called as identities. If the statement is true for some or no values of the variable, such statements are called as equations.

Example :
(i) $(x + 3)^2 = x^2 + 6x + 9$ is an identity
(ii) $(x + 3)^2 = x^2 + 6x + 8$, is an equation having no root.
(iii) $(x + 3)^2 = x^2 + 5x + 8$, is an equation having – 1 as its root.

A quadratic equation has exactly two roots which may be real (equal or unequal) or imaginary.
$ax^2 + bx + c = 0$ is:

★	a quadratic equation if	$a \neq 0$	Two Roots
★	a linear equation if	$a = 0, b \neq 0$	One Root
★	a contradiction if	$a = b = 0, c \neq 0$	No Root
★	an identity if	$a = b = c = 0$	Infinite Roots

If $ax^2 + bx + c = 0$ is satisfied by three distinct values of 'x', then it is an identity.

Example # 1 : (i) $3x^2 + 2x - 1 = 0$ is a quadratic equation here a = 3.
(ii) $(x + 1)^2 = x^2 + 2x + 1$ is an identity in x.

Solution : Here highest power of x in the given relation is 2 and this relation is satisfied by three different values x = 0, x = 1 and x = – 1 and hence it is an identity because a polynomial equation of n^{th} degree cannot have more than n distinct roots.

4. Relation Between Roots & Co-efficients:

(i) The solutions of quadratic equation, $ax^2 + bx + c = 0$, ($a \neq 0$) is given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The expression, $b^2 - 4ac \equiv D$ is called discriminant of quadratic equation.

(ii) If α, β are the roots of quadratic equation,
$ax^2 + bx + c = 0$ ....(i)
then equation (i) can be written as
$a(x - \alpha)(x - \beta) = 0$ or $ax^2 - a(\alpha + \beta)x + a\alpha\beta = 0$ ......(ii)
equations (i) and (ii) are identical,

$\therefore$ by comparing the coefficients sum of the roots, $\alpha + \beta = -\frac{b}{a} = -\frac{\text{coefficient of } x}{\text{coefficient of } x^2}$

and product of the roots, $\alpha\beta = \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of } x^2}$

(iii) Dividing the equation (i) by a, $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

$\Rightarrow$ $x^2 - \left(\frac{-b}{a}\right)x + \frac{c}{a} = 0$ $\Rightarrow$ $x^2 - (\alpha + \beta)x + \alpha\beta = 0$

$\Rightarrow$ x^2 – (sum of the roots) x + (product of the roots) = 0

Hence we conclude that the quadratic equation whose roots are α & β is $x^2 - (\alpha + \beta)x + \alpha\beta = 0$

Example # 2 : If α and β are the roots of $ax^2 + bx + c = 0$, find the equation whose roots are $\alpha+2$ and $\beta+2$.

Solution : Replacing x by x – 2 in the given equation, the required equation is
$a(x - 2)^2 + b(x - 2) + c = 0$ i.e., $ax^2 - (4a - b)x + (4a - 2b + c) = 0$.

Example # 3 : The coefficient of x in the quadratic equation $x^2 + px + q = 0$ was taken as 17 in place of 13, its roots were found to be – 2 and – 15. Find the roots of the original equation.

Solution : Here q = (– 2) × (– 15) = 30, correct value of p = 13. Hence original equation is
$x^2 + 13x + 30 = 0$ as $(x + 10)(x + 3) = 0$
$\therefore$ roots are – 10, – 3

Self practice problems :

(1) If α, β are the roots of the quadratic equation $cx^2 - 2bx + 4a = 0$ then find the quadratic equation whose roots are

(i) $\frac{\alpha}{2}, \frac{\beta}{2}$ (ii) α^2, β^2 (iii) $\alpha + 1, \beta + 1$

(iv) $\frac{1+\alpha}{1-\alpha}, \frac{1+\beta}{1-\beta}$ (v) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$

(2) If r be the ratio of the roots of the equation $ax^2 + bx + c = 0$, show that $\frac{(r+1)^2}{r} = \frac{b^2}{ac}$.

Answers : (1)
(i) $cx^2 - bx + a = 0$
(ii) $c^2x^2 + 4(b^2 - 2ac)x + 16a^2 = 0$
(iii) $cx^2 - 2x(b + c) + (4a + 2b + c) = 0$
(iv) $(c - 2b + 4a)x^2 + 2(4a - c)x + (c + 2b + 4a) = 0$
(v) $4acx^2 + 4(b^2 - 2ac)x + 4ac = 0$

5. Theory Of Equations :

If $\alpha_1, \alpha_2, \alpha_3, ...\alpha_n$ are the roots of the equation;
$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n = 0$ where $a_0, a_1, ..., a_n$ are all real & $a_0 \neq 0$ then,
$\sum \alpha_1 = -\frac{a_1}{a_0}$, $\sum \alpha_1\alpha_2 = +\frac{a_2}{a_0}$, $\sum \alpha_1\alpha_2\alpha_3 = -\frac{a_3}{a_0}$, ..., $\alpha_1\alpha_2\alpha_3.......\alpha_n = (-1)^n\frac{a_n}{a_0}$

Note : (i) If α is a root of the equation f(x) = 0, then the polynomial f(x) is exactly divisible by $(x - \alpha)$ or $(x - \alpha)$ is a factor of f(x) and conversely.

(ii) Every equation of n^{th} degree $(n \geq 1)$ has exactly n roots & if the equation has more than n roots, it is an identity.

(iii) If the coefficients of the equation f(x) = 0 are all real and $\alpha + i\beta$ is its root, then $\alpha - i\beta$ is also a root. i.e. imaginary roots occur in conjugate pairs.

(iv) An equation of odd degree will have odd number of real roots and an equation of even degree will have even numbers of real roots.

(v) If the coefficients in the equation are all rational & $\alpha + \sqrt{\beta}$ is one of its roots, then $\alpha - \sqrt{\beta}$ is also a root where $\alpha, \beta \in Q$ & β is not square of a rational number.

(vi) If there be any two real numbers 'a' & 'b' such that f(a) & f(b) are of opposite signs, then f(x) = 0 must have odd number of real roots (also atleast one real root) between 'a' and 'b'.

(vii) Every equation f(x) = 0 of degree odd has atleast one real root of a sign opposite to that of its last term. (If coefficient of highest degree term is positive).

Example # 4 : If $2x^3 + 3x^2 + 5x + 6 = 0$ has roots α, β, γ then find $\alpha + \beta + \gamma$, $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$.

Solution : Using relation between roots and coefficients, we get

$$\alpha + \beta + \gamma = = -\frac{3}{2}, \qquad \alpha\beta + \beta\gamma + \gamma\alpha = \frac{5}{2}, \qquad \alpha\beta\gamma = -\frac{6}{2} = -3.$$

Self practice problems :

(3) If $2p^3 - 9pq + 27r = 0$ then prove that the roots of the equations $rx^3 - qx^2 + px - 1 = 0$ are in H.P.

(4) If α, β, γ are the roots of the equation $x^3 + qx + r = 0$ then find the equation whose roots are

(a) $2\alpha + 2\beta + \gamma,\ \alpha + 2\beta + 2\gamma,\ 2\alpha + \beta + 2\gamma$

(b) $-\frac{r}{\alpha}, -\frac{r}{\beta}, -\frac{r}{\gamma}$

(c) $(\alpha + \beta)^2, (\beta + \gamma)^2, (\gamma + \alpha)^2$

(d) $-\alpha^3, -\beta^3, -\gamma^3$

Answers : (4) (a) $x^3 + qx - r = 0$ (b) $x^3 - qx^2 - r^2 = 0$
(c) $x^3 + 2qx^2 + q^2 x - r^2 = 0$ (d) $x^3 - 3x^2 r + (3r^2 + q^3)x - r^3 = 0$

6. Nature of Roots:

Consider the quadratic equation, $ax^2 + bx + c = 0$ having α, β as its roots;

$D \equiv b^2 - 4ac$

- $D = 0$: Roots are equal i.e. $\alpha = \beta = -b/2a$ & the quadratic expression can be expressed as a perfect square of a linear polynomial
- $D \neq 0$: Roots are unequal
  - $a, b, c \in R$ & $D > 0$: Roots are real
    - $a, b, c \in Q$ & D is square of a rational number $\Rightarrow$ Roots are rational
      - ↓ $a = 1$, $b, c \in I$ & D is square of an integer $\Rightarrow$ Roots are integral.
    - $a, b, c \in Q$ & D is not square of a rational number $\Rightarrow$ Roots are irrational i.e. $\alpha = p + \sqrt{q}$, $\beta = p - \sqrt{q}$
  - $a, b, c \in R$ & $D < 0$: Roots are imaginary $\alpha = p + iq$, $\beta = p - iq$

Example # 5 : For what values of m the equation $(1 + m)x^2 - 2(1 + 3m)x + (1 + 8m) = 0$ has equal roots.

Solution : Given equation is $(1 + m)x^2 - 2(1 + 3m)x + (1 + 8m) = 0$ ....(i)

Let D be the discriminant of equation (i).

Roots of equation (i) will be equal if D = 0.

or $4(1 + 3m)^2 - 4(1 + m)(1 + 8m) = 0$

or $4(1 + 9m^2 + 6m - 1 - 9m - 8m^2) = 0$

or $m^2 - 3m = 0$ or, $m(m - 3) = 0$

$\therefore$ $m = 0, 3$.

Example # 6 : Find all the integral values of a for which the quadratic equation $(x - a)(x - 10) + 1 = 0$ has integral roots.

Solution : Here the equation is $x^2 - (a + 10)x + 10a + 1 = 0$. Since integral roots will always be rational it means D should be a perfect square.

From (i) $D = a^2 - 20a + 96$.

$\Rightarrow \quad D = (a - 10)^2 - 4 \quad \Rightarrow \quad 4 = (a - 10)^2 - D$

If D is a perfect square it means we want difference of two perfect square as 4 which is possible only when $(a - 10)^2 = 4$ and $D = 0$.

$\Rightarrow \quad (a - 10) = \pm 2 \quad \Rightarrow \quad a = 12, 8$

Example # 7 : If the roots of the equation $(x - a)(x - b) - k = 0$ be c and d, then prove that the roots of the equation $(x - c)(x - d) + k = 0$, are a and b.

Solution : By given condition $(x - a)(x - b) - k \equiv (x - c)(x - d)$

or $(x - c)(x - d) + k \equiv (x - a)(x - b)$

Above shows that the roots of $(x - c)(x - d) + k = 0$ are a and b.

Example # 8 : Determine 'a' such that $x^2 - 11x + a$ and $x^2 - 14x + 2a$ may have a common factor.

Solution : Let $x - \alpha$ be a common factor of $x^2 - 11x + a$ and $x^2 - 14x + 2a$.

Then $x = \alpha$ will satisfy the equations $x^2 - 11x + a = 0$ and $x^2 - 14x + 2a = 0$.

$\therefore \quad \alpha^2 - 11\alpha + a = 0$ and $\quad \alpha^2 - 14\alpha + 2a = 0$

Solving (i) and (ii) by cross multiplication method, we get a = 0, 24.

Example # 9 : Show that the expression $x^2 + 2(a + b + c)x + 3(bc + ca + ab)$ will be a perfect square if a = b = c.

Solution : Given quadratic expression will be a perfect square if the discriminant of its corresponding equation is zero.

i.e. $\quad 4(a + b + c)^2 - 4.3\,(bc + ca + ab) = 0$

or $\quad (a + b + c)^2 - 3(bc + ca + ab) = 0$

or $\quad \frac{1}{2}\left((a - b)^2 + (b - c)^2 + (c - a)^2\right) = 0$

which is possible only when a = b = c.

Self practice problems :

(5) For what values of 'k' the expression $(4 - k)x^2 + 2(k + 2)x + 8k + 1$ will be a perfect square ?

(6) If $(x - \alpha)$ be a factor common to $a_1x^2 + b_1x + c$ and $a_2x^2 + b_2x + c$, then prove that $\alpha(a_1 - a_2) = b_2 - b_1$.

(7) If $3x^2 + 2\alpha xy + 2y^2 + 2ax - 4y + 1$ can be resolved into two linear factors, Prove that 'α' is a root of the equation $x^2 + 4ax + 2a^2 + 6 = 0$.

(8) Let $4x^2 - 4(\alpha - 2)x + \alpha - 2 = 0$ $(\alpha \in R)$ be a quadratic equation. Find the values of 'α' for which
 (i) Both roots are real and distinct.
 (ii) Both roots are equal.
 (iii) Both roots are imaginary
 (iv) Both roots are opposite in sign.
 (v) Both roots are equal in magnitude but opposite in sign.

(9) If $P(x) = ax^2 + bx + c$, and $Q(x) = -ax^2 + dx + c$, $ac \neq 0$ then prove that $P(x) \cdot Q(x) = 0$ has atleast two real roots.

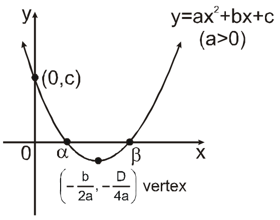
Answers. (5) 0, 3
 (8) (i) $(-\infty, 2) \cup (3, \infty)$ (ii) $\alpha \in \{2, 3\}$ (iii) $(2, 3)$ (iv) $(-\infty, 2)$ (v) ϕ

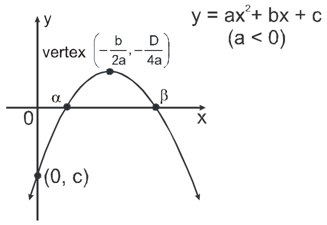
7. Graph of Quadratic Expression :

★ the graph between x, y is always a parabola.

★ the co–ordinate of vertex are $\left(-\frac{b}{2a}, -\frac{D}{4a}\right)$

★ If a > 0 then the shape of the parabola is concave upwards & if a < 0 then the shape of the parabola is concave downwards.





★ the parabola intersect the y–axis at point (0, c).

★ the x–co–ordinate of point of intersection of parabola with x–axis are the real roots of the quadratic equation f (x) = 0. Hence the parabola may or may not intersect the x–axis.

8. Range of Quadratic Expression $f(x) = ax^2 + bx + c$.

(i) Range :

If a > 0 $\Rightarrow$ $f(x) \in \left[-\frac{D}{4a}, \infty\right)$

If a < 0 $\Rightarrow$ $f(x) \in \left(-\infty, -\frac{D}{4a}\right]$

Hence maximum and minimum values of the expression f (x) is $-\frac{D}{4a}$ in respective cases and it occurs at $x = -\frac{b}{2a}$ (at vertex).

(ii) Range in restricted domain:

Given $x \in [x_1, x_2]$

(a) If $-\frac{b}{2a} \notin [x_1, x_2]$ then,

$f(x) \in [\min\{f(x_1), f(x_2)\}, \max\{f(x_1), f(x_2)\}]$

(b) If $-\frac{b}{2a} \in [x_1, x_2]$ then,

$$f(x) \in \left[\min\left\{f(x_1), f(x_2), -\frac{D}{4a}\right\}, \max\left\{f(x_1), f(x_2), -\frac{D}{4a}\right\}\right]$$

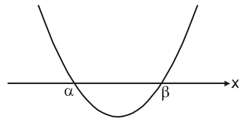
9. Sign of Quadratic Expressions :

The value of expression $f(x) = ax^2 + bx + c$ at $x = x_0$ is equal to y–co–ordinate of the point on parabola $y = ax^2 + bx + c$ whose x–co–ordinate is x_0. Hence if the point lies above the x–axis for some $x = x_0$, then $f(x_0) > 0$ and vice–versa.

We get six different positions of the graph with respect to x–axis as shown.

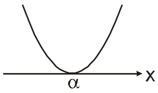
Conclusions :

(i)



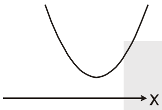
(a) $a > 0$
(b) $D > 0$
(c) Roots are real & distinct.
(d) $f(x) > 0$ in $x \in (-\infty, \alpha) \cup (\beta, \infty)$
(e) $f(x) < 0$ in $x \in (\alpha, \beta)$

(ii)



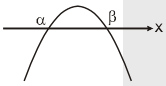
(a) $a > 0$
(b) $D = 0$
(c) Roots are real & equal.
(d) $f(x) > 0$ in $x \in R - \{\alpha\}$

(iii)



(a) $a > 0$
(b) $D < 0$
(c) Roots are imaginary.
(d) $f(x) > 0 \ \forall \ x \in R$.

(iv)



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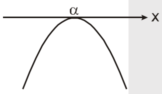

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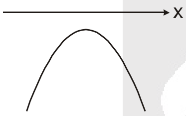


(a) $a < 0$
(b) $D < 0$
(c) Roots are imaginary.
(d) $f(x) < 0 \ \forall \ x \in R$.

Example # 10 : If $c < 0$ and $ax^2 + bx + c = 0$ does not have any real roots then prove that
(i) $a - b + c < 0$ (ii) $9a + 3b + c < 0$.

Solution : $c < 0$ and $D < 0 \Rightarrow f(x) = ax^2 + bx + c < 0$ for all $x \in R$
$\Rightarrow f(-1) = a - b + c < 0$
and $f(3) = 9a + 3b + c < 0$

Example # 11 : Find the range of $f(x) = x^2 - 5x + 6$.

Solution : minimum of $f(x) = -\dfrac{D}{4a}$ at $x = -\dfrac{b}{2a}$ $= -\left(\dfrac{25-24}{4}\right)$ at $x = \dfrac{5}{2}$ $= -\dfrac{1}{4}$

maximum of $f(x) \to \infty$

Hence range is $\left[-\dfrac{1}{4}, \infty\right)$

Example # 12 : Find the range of rational expression $y = \frac{x^2 - x + 4}{x^2 + x + 4}$ if x is real.

Solution : $y = \frac{x^2 - x + 4}{x^2 + x + 4}$ $\Rightarrow$ $(y - 1)x^2 + (y + 1)\,x + 4(y - 1) = 0$ ....(i)

case-I : if $y \neq 1$, then equation (i) is quadratic in x

and $\because$ x is real

$\therefore$ $D \geq 0$ $\Rightarrow$ $(y + 1)^2 - 16(y - 1)^2 \geq 0$ $\Rightarrow$ $(5y - 3)\,(3y - 5) \leq 0$

$\therefore$ $y \in \left[\frac{3}{5}, \frac{5}{3}\right] - \{1\}$

case-II : if y = 1, then equation becomes

$2x = 0 \Rightarrow$ x = 0 which is possible as x is real.

$\therefore$ Range $\left[\frac{3}{5}, \frac{5}{3}\right]$

Example # 13 : Find the range of $y = \frac{x + 3}{2x^2 + 3x + 9}$, if x is real.

Solution : $y = \frac{x + 3}{2x^2 + 3x + 9}$

$\Rightarrow$ $2yx^2 + (3y - 1)x + 3(3y - 1) = 0$ ....(i)

case-I : if $y \neq 0$, then equation (i) is quadratic in x

$\because$ x is real

$\therefore$ $D \geq 0$

$\Rightarrow$ $(3y - 1)^2 - 24y\,(3y - 1) \geq 0$

$\Rightarrow$ $(3y - 1)\,(21y + 1) \leq 0$

$y \in \left[-\frac{1}{21}, \frac{1}{3}\right] - \{0\}$

case-II : if y = 0, then equation becomes

x = –3 which is possible as x is real

$\therefore$ Range $y \in \left[-\frac{1}{21}, \frac{1}{3}\right]$

Self practice problems :

(10) If c > 0 and $ax^2 + 2bx + 3c = 0$ does not have any real roots then prove that

(i) $4a - 4b + 3c > 0$ (ii) $a + 6b + 27c > 0$ (iii) $a + 2b + 6c > 0$

(11) If $f(x) = (x - a)\,(x - b)$, then show that $f(x) \geq -\frac{(a - b)^2}{4}$.

(12) Find the least integral value of 'k' for which the quadratic polynomial $(k - 1)\,x^2 + 8x + k + 5 > 0 \ \forall \ x \in R$.

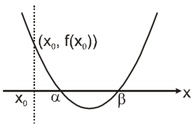
(13) Find the range of the expression $\frac{x^2 + 34x - 71}{x^2 + 2x - 7}$, if x is a real.

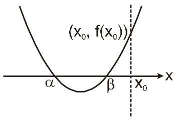
(14) Find the interval in which 'm' lies so that the expression $\frac{mx^2 + 3x - 4}{-4x^2 + 3x + m}$ can take all real values, $x \in R$.

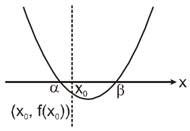
Answers : (12) k = 4 (13) $(-\infty, 5] \cup [9, \infty)$ (14) $m \in (1, 7)$

10. Location of Roots :

Let $f(x) = ax^2 + bx + c$, where $a > 0$ & $a, b, c \in R$.

(i) 

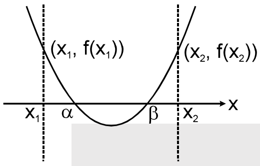
(ii) 

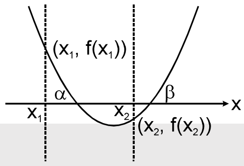
(iii) 

(i) Conditions for both the roots of f (x) = 0 to be greater than a specified number 'x_0' are $b^2 - 4ac \geq 0$ & $f(x_0) > 0$ & $(-b/2a) > x_0$.

(ii) Conditions for both the roots of f (x) = 0 to be smaller than a specified number 'x_0' are $b^2 - 4ac \geq 0$ & $f(x_0) > 0$ & $(-b/2a) < x_0$.

(iii) Conditions for a number 'x_0' to lie between the roots of f (x) = 0 is $f(x_0) < 0$.

(iv) 

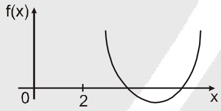
(v) 

(iv) Conditions that both roots of f (x) = 0 to be confined between the numbers x_1 and x_2, $(x_1 < x_2)$ are $b^2 - 4ac \geq 0$ & $f(x_1) > 0$ & $f(x_2) > 0$ & $x_1 < (-b/2a) < x_2$.

(v) Conditions for exactly one root of f (x) = 0 to lie in the interval (x_1, x_2) i.e. $x_1 < x < x_2$ is $f(x_1).f(x_2) < 0$.

Example # 14 : Let $x^2 - (m - 3)x + m = 0$ $(m \in R)$ be a quadratic equation, then find the values of 'm' for which

(a) both the roots are greater than 2.
(b) both roots are positive.
(c) one root is positive and other is negative.
(d) One root is greater than 2 and other smaller than 1
(e) Roots are equal in magnitude and opposite in sign.
(f) both roots lie in the interval (1, 2)

Solution : (a) 

Condition - I : $D \geq 0$ $\Rightarrow$ $(m - 3)^2 - 4m \geq 0$ $\Rightarrow$ $m^2 - 10m + 9 \geq 0$

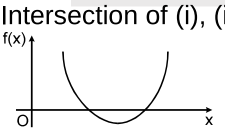
$\Rightarrow$ $(m - 1)(m - 9) \geq 0$

$\Rightarrow$ $m \in (-\infty, 1] \cup [9, \infty)$ ......(i)

Condition - II : $f(2) > 0$ $\Rightarrow$ $4 - (m - 3)2 + m > 0 \Rightarrow$ $m < 10$ ......(ii)

Condition - III : $-\dfrac{b}{2a} > 2$ $\Rightarrow$ $\dfrac{m-3}{2} > 2 \Rightarrow$ $m > 7$ ......(iii)

Intersection of (i), (ii) and (iii) gives $m \in [9, 10)$

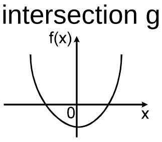
(b) 

Condition - I $D \geq 0$ $\Rightarrow$ $m \in (-\infty, 1] \cup [9, \infty)$

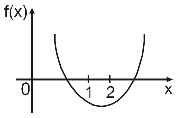
Condition - II $f(0) > 0$ $\Rightarrow$ $m > 0$

Condition - III $-\dfrac{b}{2a} > 0$ $\Rightarrow$ $\dfrac{m-3}{2} > 0$ $\Rightarrow$ $m > 3$

intersection gives $m \in [9, \infty)$ Ans.

(c) 

Condition - I $f(0) < 0$ $\Rightarrow$ $m < 0$ Ans.

(d) 

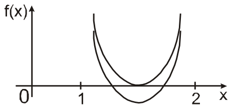
Condition - I $f(1) < 0 \Rightarrow 4 < 0 \Rightarrow m \in \phi$

Condition - II $f(2) < 0 \Rightarrow m > 10$

Intersection gives $m \in \phi$ Ans.

(e) sum of roots = 0 $\Rightarrow m = 3$

and $f(0) < 0 \Rightarrow m < 0 \quad \therefore \quad m \in \phi$ **Ans.**

(f) 

Condition - I $D \geq 0 \Rightarrow m \in (-\infty, 1] \cup [9, \infty)$

Condition - II $f(1) > 0 \Rightarrow 1 - (m - 3) + m > 0 \Rightarrow 4 > 0$ which is true $\forall\, m \in R$

Condition - III $f(2) > 0 \Rightarrow m < 10$

Condition - IV $1 < -\frac{b}{2a} < 2 \Rightarrow 1 < \frac{m-3}{2} < 2 \Rightarrow 5 < m < 7$

intersection gives $m \in \phi$ **Ans.**

Example # 15 : Find all the values of 'a' for which both the roots of the equation $(a - 2)x^2 - 2ax + a = 0$ lies in the interval (– 2, 1).

Solution : Case-I : $f(-2) > 0 \Rightarrow 4(a - 2) + 4a + a > 0$

$9a - 8 > 0 \Rightarrow a > \frac{8}{9}$

$f(1) > 0 \Rightarrow a - 2 - 2a + a > 0$

$-2 > 0$ not possible $\quad \therefore \quad a \in \phi$

Case-II : $a - 2 < 0 \Rightarrow a < 2$

$f(-2) < 0 \Rightarrow a < \frac{8}{9}$

$f(1) < 0 \Rightarrow a \in R$

$-2 < \frac{b}{2a} - < 1 \Rightarrow a < \frac{4}{3}$

$D \geq 0 \Rightarrow a \geq 0$

intersection gives $a \in \left[0, \frac{8}{9}\right)$

complete solution $a \in \left[0, \frac{8}{9}\right) \cup \{2\}$

Self practice problems :

(15) Let $x^2 - 2(a - 1)x + a - 1 = 0$ $(a \in R)$ be a quadratic equation, then find the value of 'a' for which
(a) Both the roots are positive
(b) Both the roots are negative
(c) Both the roots are opposite in sign.
(d) Both the roots are greater than 1.
(e) Both the roots are smaller than 1.
(f) One root is small than 1 and the other root is greater than 1.

(16) Find the values of p for which both the roots of the equation $4x^2 - 20px + (25p^2 + 15p - 66) = 0$ are less than 2.

(17) Find the values of 'α' for which 6 lies between the roots of the equation $x^2 + 2(\alpha - 3)x + 9 = 0$.

(18) Let $x^2 - 2(a - 1)x + a - 1 = 0$ $(a \in R)$ be a quadratic equation, then find the values of 'a' for which
(i) Exactly one root lies in (0, 1).
(ii) Both roots lies in (0, 1).
(iii) Atleast one root lies in (0, 1).
(iv) One root is greater than 1 and other root is smaller than 0.

(19) Find the values of a, for which the quadratic expression $ax^2 + (a - 2)x - 2$ is negative for exactly two integral values of x.

Answers : (15) (a) $[2, \infty)$ (b) ϕ (c) $(-\infty, 1)$ (d) ϕ (e) $(-\infty, 1]$ (f) $(2, \infty)$

(16) $(-\infty, -1)$ (17) $\left(-\infty, -\frac{3}{4}\right)$

(18) (i) $(-\infty, 1) \cup (2, \infty)$ (ii) ϕ (iii) $(-\infty, 1) \cup (2, \infty)$ (iv) ϕ

(19) $[1, 2)$

11. Common Roots:

Consider two quadratic equations, $a_1x^2 + b_1x + c_1 = 0$ & $a_2x^2 + b_2x + c_2 = 0$.

(i) If two quadratic equations have both roots common, then the equations are identical and their co-efficient are in proportion.

i.e. $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

(ii) If only one root is common, then the common root 'α' will be :

$\alpha = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1} = \frac{b_1c_2 - b_2c_1}{c_1a_2 - c_2a_1}$

Hence the condition for one common root is :

$\Rightarrow \quad (c_1a_2 - c_2a_1)^2 = (a_1b_2 - a_2b_1)(b_1c_2 - b_2c_1)$

Note : If f(x) = 0 & g(x) = 0 are two polynomial equation having some common root(s) then those common root(s) is/are also the root(s) of $h(x) \equiv a f(x) + bg(x) = 0$.

Example # 16 : If $x^2 - ax + b = 0$ and $x^2 - px + q = 0$ have a root in common and the second equation has equal roots, show that $b + q = \frac{ap}{2}$.

Solution : Given equations are : $x^2 - ax + b = 0$ .... (i)

and $x^2 - px + q = 0.$ .... (ii)

Let α be the common root. Then roots of equation (ii) will be α and α. Let β be the other root of equation (i). Thus roots of equation (i) are α, β and those of equation (ii) are α, α.

Now $\alpha + \beta = a$ .... (iii)

$\alpha\beta = b$ .... (iv)

$2\alpha = p$ .... (v)

$\alpha^2 = q$ .... (vi)

L.H.S. $= b + q = \alpha\beta + \alpha^2 = \alpha(\alpha + \beta)$ .... (vii)

and R.H.S. $= \frac{ap}{2} = \frac{(\alpha + \beta)\,2\alpha}{2} = \alpha(\alpha + \beta)$ .... (viii)

from (vii) and (viii), L.H.S. = R.H.S.

Example # 17 : If a, b, c $\in$ R and equations $ax^2 + bx + c = 0$ and $x^2 + 2x + 9 = 0$ have a common root, show that a : b : c = 1 : 2 : 9.

Solution : Given equations are : $x^2 + 2x + 9 = 0$ ....(i)

and $ax^2 + bx + c = 0$ ....(ii)

Clearly roots of equation (i) are imaginary since equation (i) and (ii) have a common root, therefore common root must be imaginary and hence both roots will be common.

Therefore equations (i) and (ii) are identical

$\therefore \quad \frac{a}{1} = \frac{b}{2} = \frac{c}{9}$

$\therefore \quad a : b : c = 1 : 2 : 9$

Self practice problems :

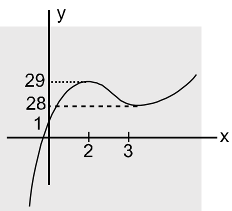
(20) If the equations $ax^2 + bx + c = 0$ and $x^3 + x - 2 = 0$ have two common roots then show that $2a = 2b = c$.

(21) If $ax^2 + 2bx + c = 0$ and $a_1x^2 + 2b_1x + c_1 = 0$ have a common root and $\frac{a}{a_1}, \frac{b}{b_1}, \frac{c}{c_1}$ are in A.P. show that a_1, b_1, c_1 are in G.P.

12. Graphs of Polynomials

$y = a_nx^n + ... + a_1x + a_0$. The points where $y' = 0$ are called turning points which are critical in plotting the graph.

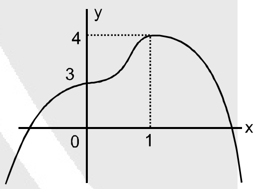
Example # 18 : Draw the graph of $y = 2x^3 - 15x^2 + 36x + 1$



Solution. $y' = 6x^2 - 30x + 36 = 6(x - 3)(x - 2)$

x	2	3	∞	–∞
y	29	28	∞	–∞

Example # 19 : Draw the graph of $y = -3x^4 + 4x^3 + 3$,



Solution. $y' = -12x^3 + 12x$

$y' = -12x^2 (x - 1)$

x	0	1	∞	–∞
y	3	4	–∞	–∞