High Level Problems (HLP)

1. Find the number of values of x satisfying the relation

$$\alpha_1^3\left(\frac{\prod_{i=2}^{n}(x-\alpha_i)}{\prod_{i=2}^{n}(\alpha_1-\alpha_i)}\right)+\sum_{j=2}^{n-1}\left(\left(\frac{\prod_{i=1}^{j-1}(x-\alpha_i)\prod_{i=j+1}^{n}(x-\alpha_i)}{\prod_{i=1}^{j-1}(\alpha_j-\alpha_i)\prod_{i=j+1}^{n}(\alpha_j-\alpha_i)}\right)\alpha_j^3\right)+\left(\frac{\prod_{i=1}^{n-1}(x-\alpha_i)}{\prod_{i=1}^{n-1}(\alpha_n-\alpha_i)}\right)\alpha_n^3-x^3=0 \text{ (where } n\ge 5).$$

2. Prove that roots of $a^2x^2 + (b^2 + a^2 - c^2)x + b^2 = 0$ are not real, if $a + b > c$ and $|a - b| < c$. (where a, b, c are positive real numbers)

3. Solve the inequality, $\frac{1}{x-1}-\frac{4}{x-2}+\frac{4}{x-3}-\frac{1}{x-4}<\frac{1}{30}$.

4. If three real and distinct numbers a, b, c are in G.P. (i.e., $b^2 = ac$) and $a + b + c = xb$, then prove that $x < -1$ or $x > 3$.

5. If $V_n = \alpha^n + \beta^n$, where α, β are roots of equation $x^2 + x - 1 = 0$. Then prove that $V_n + V_{n-3} = 2V_{n-2}$ and hence evaluate V_7 (n is a whole number)

6. Find all 'm' for which $f(x) \equiv x^2 - (m - 3)x + m > 0$ for all values of 'x' in [1, 2].

7. Find the values of a, for which the quadratic expression $ax^2 + (a - 2)x - 2$ is negative for exactly two integral values of x.

8. Find the number of real roots of $\left(x+\frac{1}{x}\right)^3+\left(x+\frac{1}{x}\right)=0$

9. If α, β are roots of the equation $x^2 - 34x + 1 = 0$, evaluate $\sqrt[4]{\alpha}-\sqrt[4]{\beta}$, where $\sqrt[4]{\ .}$ denotes the principal value.

10. Find the values of 'a' for which the equation
$(x^2+x+2)^2-(a-3)(x^2+x+2)(x^2+x+1)+(a-4)(x^2+x+1)^2=0$ has atleast one real root.

11. Show that the quadratic equation $x^2 + 7x - 14(q^2 + 1) = 0$ where q is an integer, has no integral roots.

12. Find the integral values of 'a' for which the equation $x^4-(a^2-5a+6)x^2-(a^2-3a+2)=0$ has only real roots.

13. If α, β; β, γ and γ, α are the roots of $a_ix^2+b_ix+c_i=0$; i = 1, 2, 3 then show that

$$(\alpha+\beta+\gamma)+(\alpha\beta+\beta\gamma+\alpha\gamma)+\alpha\beta\gamma=\pm\left\{\prod_{i=1}^{3}\left(\frac{a_i-b_i+c_i}{a_i}\right)\right\}^{1/2}-1$$

14. Suppose that $a_1 > a_2 > a_3 > a_4 > a_5 > a_6$ and

$p = a_1 + a_2 + a_3 + a_4 + a_5 + a_6$

$q = a_1a_3 + a_3a_5 + a_5a_1 + a_2a_4 + a_4a_6 + a_6a_2$

$r = a_1a_3a_5 + a_2a_4a_6$,

then show that roots of the equation $2x^3 - px^2 + qx - r = 0$ are real.

15. If $\beta + \cos^2\alpha$, $\beta + \sin^2\alpha$ are the roots of $x^2 + 2bx + c = 0$ and $\gamma + \cos^4\alpha$, $\gamma + \sin^4\alpha$ are the roots of $X^2 + 2BX + C = 0$, then prove that $b^2 - B^2 = c - C$.

16. Find the set of values of 'a' if $(x^2 + x)^2 + a(x^2 + x) + 4 = 0$ has
(i) all four real & distinct roots.
(ii) four roots in which only two roots are real and distinct.
(iii) all four imaginary roots.
(iv) four real roots in which only two are equal.

17. $f(x) = x^2 + bx + c$, where $b, c \in R$, if f(x) is a factor of both $x^4 + 6x^2 + 25$ and $3x^4 + 4x^2 + 28x + 5$ then find f(x).

18. Let $ax^4 + bx^3 + x^2 + (3-a)x + 3 = 0$ and $x^2 + (2-a)x + 3 = 0$ have common roots. If $a \in (-1,5)$ then find $|a+12b|$

19. How many quadratic equations are there which are unchanged by squaring their roots ?

20. Let $P(x) = x^5 + x^2 + 1$ have zeros $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ and $Q(x) = x^2 - 2$, then find

(i) $\prod_{i=1}^{5} Q(\alpha_i)$ (ii) $\sum_{i=1}^{5} Q(\alpha_i)$ (iii) $\sum_{1 \le i < j \le 5}\sum Q(\alpha_i)\,Q(\alpha_j)$ (iv) $\sum_{i=1}^{5} Q^2(\alpha_i)$

21. If a, b, c are non–zero, unequal rational numbers then prove that the roots of the equation $(abc^2)x^2 + 3a^2cx + b^2cx - 6a^2 - ab + 2b^2 = 0$ are rational.

22. If a, b, c represents sides of a Δ then prove that equation $x^2 - (a^2 + b^2 + c^2)x + a^2b^2 + b^2c^2 + c^2a^2 = 0$ has imaginary roots.

23. If x_1 is a root of $ax^2 + bx + c = 0$, x_2 is a root of $-ax^2 + bx + c = 0$ where $0 < x_1 < x_2$, show that the equation $ax^2 + 2bx + 2c = 0$ has a root x_3 satisfying $0 < x_1 < x_3 < x_2$.

24. Find the number of positive real roots of $x^4 - 4x - 1 = 0$.

25. If $(1 + k)\tan^2 x - 4\tan x - 1 + k = 0$ has real roots $\tan x_1$ and $\tan x_2$, where $\tan x_1 \neq \tan x_2$, then find k.

26. Let Δ^2 be the discriminant and α, β be the roots of the equation $ax^2 + bx + c = 0$. Then find equation whose roots are $2a\alpha + \Delta$ and $2a\beta - \Delta$.

27. Prove that $\dfrac{\pi^e}{x-e} + \dfrac{e^\pi}{x-\pi} + \dfrac{\pi^\pi + e^e}{x-\pi-e} = 0$ has one real root in (e, π) and other in $(\pi, \pi + e)$.

28. If α, β^2 are integers, β^2 is non-zero multiple of 3 and $\alpha + i\beta$, -2α are roots of $x^3 + ax^2 + bx - 316 = 0$, $a, b, \beta \in R$, then find a, b.

29. Let polynomial $f(x) = ax^4 + bx^3 + cx^2 + dx + e$ have integral coefficient (where $a > 0$) If there exist four distinct integer $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ ($\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$) such that $f(\alpha_1) = f(\alpha_2) = f(\alpha_3) = f(\alpha_4) = 5$ and equation $f(x) = 9$ has atleast one integeral roots then find

(i) $f\left(\dfrac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}{4}\right)$ (ii) $f'\left(\dfrac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4}{4}\right)$

(iii) Range of f(x) in $[\alpha_2, \alpha_3]$

(iv) Difference of largest and smallest root of equation f(x) = 9

30. If x and y both are non-negative integral values for which $(xy - 7)^2 = x^2 + y^2$, then find the sum of all possible values of x.

31. Find the set of all real values of λ such that the root of the equation
$x^2 + 2(a + b + c)x + 3\lambda(ab + bc + ca) = 0$ are always real for any choice of a, b, c (where a, b, c represents sides of scalene triangle).

(A) $\left(-\infty, \frac{4}{3}\right)$ (B) $\left(\frac{4}{3}, \infty\right)$ (C) $\left(\frac{1}{3}, \frac{5}{3}\right)$ (D) $\left(\frac{4}{3}, \frac{5}{3}\right)$

32_. Let $P(x) = x^2 + bx + c$ $(b, c \in R)$, then which of the following statement implies that $P(P(x)) = 0$ has atleast one negative root.

(A) P(x) = 0 has root of opposite sign
(B) P(x) = 0 has both roots positive
(C) P(x) = 0 has both roots negative
(D) $\left(c - \frac{b^2}{4}\right)^2 + b\left(c - \frac{b^2}{4}\right) + c < 0$ & $b > 0$

Answers

1. Infinite
3. $(-\infty, -2) \cup (-1, 1) \cup (2, 3) \cup (4, 6) \cup (7, \infty)$
5. –29
6. $(-\infty, 10)$
7. [1, 2)
8. 0
9. ± 2
10. $5 < a \le \frac{19}{3}$
12. $a \in \{1, 2\}$
16. (i) $a \in (-\infty, -4)$ (ii) $a \in \left(\frac{65}{4}, \infty\right)$ (iii) $a \in \left(-4, \frac{65}{4}\right)$ (iv) $a \in \phi$
17. $x^2 - 2x + 5$
18. 3
19. 4
20. (i) – 23 (ii) – 10 (iii) 40 (iv) 20
24. 1
25. $\left(-\sqrt{5}, -1\right) \cup \left(-1, \sqrt{5}\right)$
26. $x^2 + 2bx + b^2 = 0$ or $x^2 + 2bx - 3b^2 + 16ac = 0$
28. a = 0, b = 63
29. (i) 9 (ii) 0 (iii) [5, 9] (iv) $2\sqrt{5}$
30. 14
31. (A)
32. (AD)