SOLUTIONS OF GRAVITATION

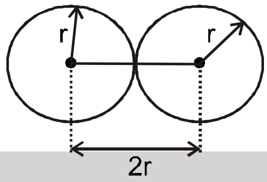
EXERCISE-1

PART-I

A-1. $F = G\dfrac{m_1m_2}{r^2} = 6.67 \times 10^{-11} \times \dfrac{1.4\times 7.34\times 10^{22}}{(378\times 10^6)^2} = 4.8 \times 10^{-5}$ N

A-2. mass of each sphere

m = Volume × ρ

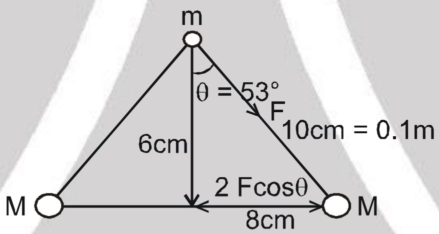


$\dfrac{4}{3} = \pi r^3\rho$

$F = G\dfrac{m.m}{(2r)^2} = \dfrac{G\left(\dfrac{4}{3}\pi r^3\rho\right)^2}{4r^2} = \dfrac{4}{9}G\pi^2\rho^2r^4$ N **Ans.**

A-3. $\tan\theta = \dfrac{8}{6} = \dfrac{4}{3}$

$\theta = 53°$



$F = \dfrac{GmM}{r^2} = G\dfrac{0.260\times 0.01}{(0.1)^2}$

$a = \dfrac{2F\cos\theta}{m} = 2G\dfrac{0.260}{(0.1)^2}\left(\dfrac{3}{5}\right) = 31.2$ G m/s^2

B-1. $E_x = \dfrac{\partial v}{\partial x} - \dfrac{\partial}{\partial x} = -(20x + 40y) = -20$

$E_y = -\dfrac{\partial v}{\partial x} = -\dfrac{\partial}{\partial y}(20x + 40y) = -40$

$\vec{E} = E_x\hat{i} + E_y\hat{j} = -20\hat{i} - 40\hat{j}$

It is independent of co ordinates

Force $= \vec{F} = m\vec{E} = 0.25\{-20\hat{i} - 40\hat{j}\} = -5\hat{i} - 10\hat{j}$

magnitude of $\vec{F} = \sqrt{5^2 + 10^2} = 5\sqrt{5}$ N

B-2. $V = -\dfrac{GM}{R} = -\dfrac{G\rho\dfrac{4}{3}\pi R^3}{R} = -\dfrac{4}{3}\pi G\rho R^2$

C-1. Potential energy at ground surface

potential energy = $\frac{-GMm}{R}$

potential energy at a height of R is

potential energy = $\frac{-GMm}{2R}$

When a body comes to ground

Loss in potential energy = Gain in kinetic energy

$\Rightarrow \quad \frac{-GMm}{2R} - \left(\frac{-GMm}{R}\right) = \frac{1}{2} mv^2$

$\Rightarrow \quad \frac{GMm}{2R} = \frac{1}{2} mv^2$

$\Rightarrow \quad gR = v^2 \qquad \left(\because \frac{GM}{R^2} = g\right)$

$\Rightarrow \quad v = \sqrt{gR}$

C-2. Initial kinetic energy = $\frac{1}{2} M_S V^2$

Initial potential energy = $-\frac{GM_A M_S}{d/2} - \frac{GM_B M_S}{d/2} = -\frac{2GM_S}{d}(M_A + M_B)$

Total initial energy = $\frac{1}{2} M_S V^2 - \frac{2GM_S}{d}(M_A + M_B)$

Finally, Potential energy = 0

Kinetic energy = 0

$\Rightarrow$ Limiting case

Applying energy cnservation

$$M_S V^2 - \frac{2GM_S}{d}(M_A + M_B) = 0 \Rightarrow V = 2\sqrt{\frac{G(M_A + M_B)}{d}}$$

D-1. $T_1 = 2\pi\sqrt{\frac{r^3}{GM_e}}$, $T_2 = 2\pi\sqrt{\frac{(1.01r)^3}{GM_e}}$

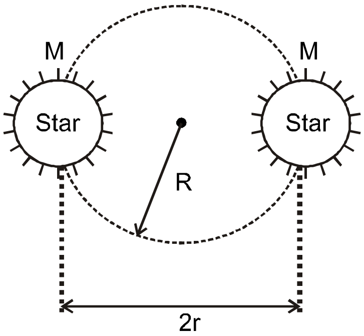
$\frac{T_2}{T_1} = \left(\frac{1.01r}{r}\right)^{3/2}$

$\frac{T_2}{T_1} = [1 + 0.01]^{3/2} = 1 + \frac{3}{2} \times 0.01$

$\frac{T_2}{T_1} - 1 = 0.005 \times 3$

$\frac{(T_2 - T_1)}{T_1} \times 100 = 0.015 \times 100 = 1.5\%.$

D-2. **(a)** $F = \dfrac{GMm}{(2R)^2} = \dfrac{GM^2}{4R^2}$



(b) $\dfrac{Mv^2}{R} = \dfrac{GM^2}{4R^2}$

$v = \sqrt{\dfrac{GM}{4R}}$

$T = \dfrac{2\pi R}{v} = \dfrac{2\pi R}{\sqrt{\dfrac{GM}{4R}}} = 4\pi\sqrt{\dfrac{R^3}{GM}}$

(c) Angular speed:

$\omega = \dfrac{2\pi}{T} = \dfrac{2\pi}{4\pi\left(\sqrt{\dfrac{R^3}{GM}}\right)} = \sqrt{\dfrac{GM}{4R^3}}$

(d) Energy required to separate = – {total energy}

= – {Kinetic energy + Potential energy}

$= -\left\{\dfrac{1}{2}Mv^2 + \dfrac{1}{2}Mv^2 - \dfrac{GM^2}{2R}\right\} = -\left\{Mv^2 - \dfrac{GM^2}{2R}\right\}$

$= -\left\{M\dfrac{GM}{4R} - \dfrac{GM^2}{2R}\right\} = -\left\{-\dfrac{GM^2}{4R}\right\} = \dfrac{GM^2}{4R}$

(e) Let its velocity = 'v'

Kinetic energy = $\dfrac{1}{2}mv^2$

Potential at centre of mass = $-\dfrac{GM}{R} - \dfrac{GM}{R} = -\dfrac{2GM}{R}$

Potential energy at centre of mass = $-\dfrac{2GMm}{R}$

For particle to reach infinity:

Kinetic energy + Potential energy = 0

$\dfrac{1}{2}mv^2 \times \dfrac{2GMm}{R} = 0$

$v = \sqrt{\dfrac{4GM}{R}}$ **Ans.**

D-3. (a) $\frac{U_A}{U_B} = \frac{\frac{-GMm_A}{r_A}}{\frac{-GMm_B}{r_B}} = \frac{m_A}{m_B}\frac{r_B}{r_A}$

$r_B = 19200 + 6400 = 25600$ Km

$r_A = 6400 + 6400 = 12800$ km, $m_A = m_B$

$\frac{U_A}{U_B} = \frac{25600}{12800} = 2$

(b) $\frac{K_A}{K_B} = \frac{\frac{GMm_A}{2r_A}}{\frac{GMm_B}{2r_B}} = \frac{m_A}{m_B}\frac{r_B}{r_A} = 2$

(c) As T.E. $= -\frac{GMm}{2r}$,

Clearly farther the satellite from the earth, the greater is its total energy. Thus B is having more energy.

D-4. $T^2 \propto r^3$

For the Saturn, r is more, So T will be more as $\omega = \frac{2\pi}{T}$ so ω will be less, for the Saturn.

and orbital speed $V_o = \sqrt{\frac{GM}{r}}$

as r is more So V_O will be less for the Saturn.

E-1. Field outside shell $g^1 = g\left(1-\frac{2h}{R}\right)$

So $9 = g\left(1-\frac{2h}{R}\right) \Rightarrow \frac{h}{R} = \frac{1}{20}$

Field inside shell $E = g\left(1-\frac{h}{R}\right)$

$E = 10\left(1-\frac{1}{20}\right)$ $\quad E = 10 - 0.5 = 9.5$ m/sec^2

Solving we get $x \approx 9.5$ m/s^2

E-2. Period of pendulum $= 2\pi\sqrt{\frac{\ell}{g}}$

Let T_1 be the time period at pole and T_2 is time period at equator.

$\frac{T_1}{T_2} = \sqrt{\frac{g_2}{g_1}}$

$\frac{T_1}{1} = \sqrt{\frac{g\left(1-\frac{R_e\omega^2}{g}\right)}{g}} = \left(1-\frac{R_e\omega^2}{g}\right)^{\frac{1}{2}}$

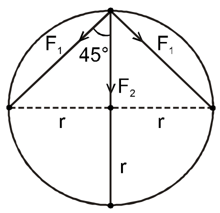
$T_1 = 1 - \frac{R_e\omega^2}{2g}$. Since $\frac{R_e\omega^2}{g} << 1$

So $T_1 = 1 - \frac{1}{2}\frac{R_e\omega^2}{g} = 1 - \frac{1}{2}\frac{(2\pi)^2}{(86400)^2} \times \frac{6400\times10^3}{9.8} = 0.998$ second **Ans.**

PART-II

A-1 Net force is towards centre

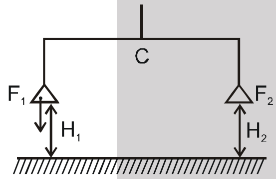
$$F_{net} = F_2 + \frac{2F_1}{\sqrt{2}} = F_2 + \sqrt{2}\,F_1$$



$$\left.\begin{aligned} F_2 &= \frac{Gm^2}{(2r)^2} = \frac{Gm^2}{4r^2} \\ F_1 &= \frac{Gm^2}{(\sqrt{2}r)^2} = \frac{Gm^2}{2r^2} \end{aligned}\right\} \rightarrow F_{net} = \frac{Gm^2}{4r^2} + \frac{\sqrt{2}Gm^2}{2r^2} = \frac{mv^2}{r}$$

$$\Rightarrow v = \sqrt{\frac{Gm}{4r}(1+2\sqrt{2})}$$

A-2.



Net torque $= F_2.\frac{\ell}{2} - F_1.\frac{\ell}{2}$

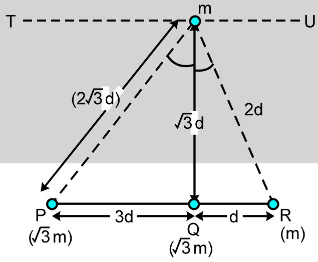
$= (F_2 - F_1)\,\frac{\ell}{2}$

$F_2 = mg_{H2} = mg\left\{1 - \frac{2H_2}{R}\right\}$

$F_1 = mg_{H1} = mg\left\{1 - \frac{2H_1}{R}\right\}$

$\tau = (F_2 - F_1)\,\frac{\ell}{2} = \frac{mg\,(H_1 - H_2)\,\ell}{R}$.

A-3. In horizontal direction



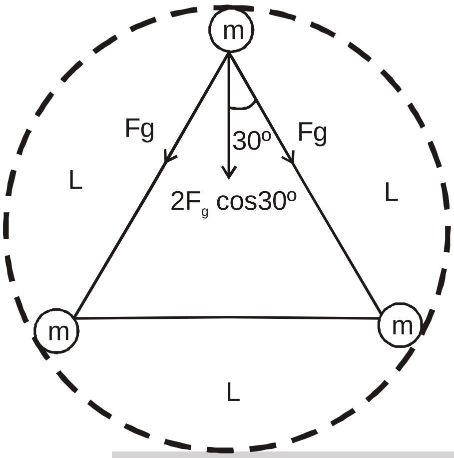
Net force $= \frac{G\sqrt{3}mm}{12d^2}\cos30° - \frac{Gm^2}{4d^2}\cos60° = \frac{Gm^2}{8d^2} - \frac{Gm^2}{8d^2} = 0$

in vertical direction

Net force $= \frac{G\sqrt{3}m^2}{12d^2}\cos60° + \frac{G\sqrt{3}m^2}{3d^2} + \frac{Gm^2}{4d^2}\cos30°$

$= \frac{\sqrt{3}Gm^2}{24d^2} + \frac{\sqrt{3}Gm^2}{3d^2} + \frac{\sqrt{3}Gm^2}{8d^2} = \frac{\sqrt{3}Gm^2}{d^2}\left[\frac{1+8+3}{24}\right] = \frac{\sqrt{3}Gm^2}{2d^2}$ along SQ

A-4.



$$2\,F_g \cos 30 = \frac{MV^2}{R}$$

$$2\left(\frac{GM^2}{L^2}\right)\frac{\sqrt{3}}{2} = \frac{MV^2}{L/\sqrt{3}}$$

$$V = \sqrt{\frac{GM}{L}}$$

B-1. $dv = -\,Edr$

$$= \frac{k}{r}\,dr$$

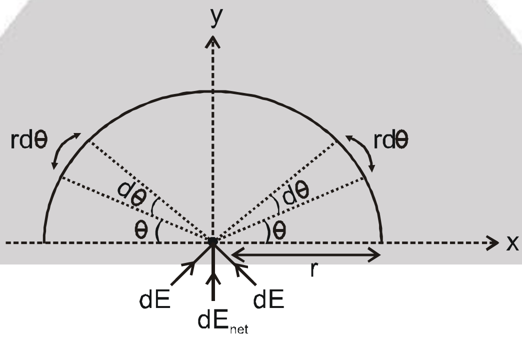
Integrating both sides

$$[v]_{v_i}^{v} = k[\ell nr]_{d_i}^{r}$$

$$v - v_i = k\ell n\frac{r}{d_i}$$

$$v = v_i + k\ell n\,\frac{r}{d_i} \quad \textbf{Ans.}$$

B-2. $dE_{net} = 2dE\,\sin\theta$



$$= \frac{Gdm}{r^2}\;2\,\sin\theta = 2G.\,\frac{\lambda rd\theta}{r^2}\,\sin\theta = \frac{2G\lambda}{r}\,\sin\theta\,d\theta$$

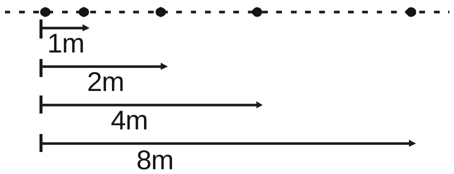
$$E_{net} = \int dE_{net} = \int_0^{\pi/2}\frac{2G\lambda}{r}\sin\theta\;d\theta = \frac{2G\lambda}{r}$$

$$\lambda = \frac{m}{\ell} \text{ and } r = \frac{\ell}{\pi}$$

$$E_{net} = \frac{2Gm\pi}{\ell^2}$$

Along + y-axis

B-3.



$v = v_1 + v_2 + v_3 + v_4 + ...$

$= -\dfrac{Gm}{1} - \dfrac{Gm}{2} - \dfrac{Gm}{4} - \dfrac{Gm}{8} - \dfrac{Gm}{16} - \dfrac{Gm}{32} - ....$

$= -Gm\left(1 + \dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + ...\right) \rightarrow$ G.P. of infinite series

$= -m\left(\dfrac{1}{1 - 1/2}\right)$; $v = -Gm\,(2) = -2Gm$ **Ans.**

B-4. For point 'A' :
For any point outside, the shells acts as point situated at centre.

So, $F_A = \dfrac{G(M_1 + M_2)}{p^2}\, m$

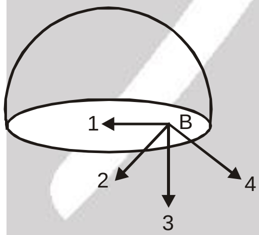
For point 'B' :
There will be no force by shell B.

So, $F_B = \dfrac{GM_1 m}{q^2}$

For point 'C' :
There will be no gravitational field.
So, $F_C = 0$

B-5. If we take complete spherical shell than gravitational field intensity at P will be zero hence for the hemi spherical shell shown the intensity at P will be along c.

B-6. Let the possible direction of gravitational field at point B be shown by 1, 2, 3 and 4(Figure 1). Rotate the figure upside down. It will be as shown in figure 2.

Figure 1 

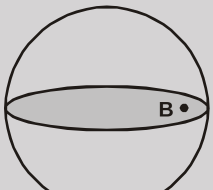
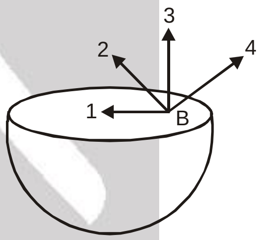
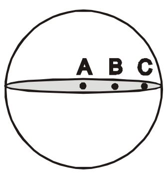
= 

Figure 3 ; Figure 2 

Now on placing upper half of figure 1 on the lower half of figure 2 we get complete sphere. Gravitational field at point B must be zero, which is only possible if the gravitational field is along direction 3. Hence gravitational field at all points on circular base of hemisphere is normal to plane of circular base.

$\therefore$ Circular base of hemisphere is an equipotential surface.

Aliter : Consider a shaded circle which divides a uniformly thin spherical shell into two equal halves.The potential at points A,B and C lying on the shaded circle is same. The potential at all these points due to upper hemisphere is half that due to complete sphere.Hence potential at points A,B and Cis also same due to upper hemispehre



C-1. Initial total energy = Initial kinetic energy + initial potential energy

$$= \frac{1}{2} m (0)^2 + \left(-\frac{GMm}{R_0}\right) = -\frac{GMm}{R_0}$$

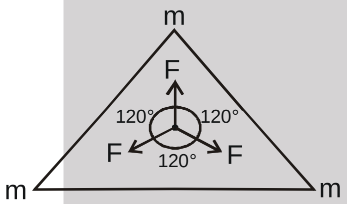
Total energy, when it reaches the surface of earth = $\frac{1}{2} mv^2 + \left(-\frac{GMm}{R}\right)$

Applying energy conservation,

$$\frac{1}{2} mv^2 - \frac{GMm}{R} = -\frac{GMm}{R_0}$$

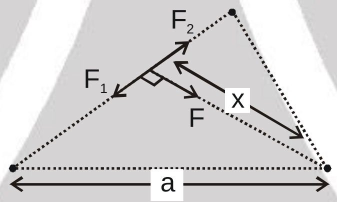
$$v = \sqrt{2GM\left\{\frac{1}{R} - \frac{1}{R_0}\right\}}$$

C-2. (a)



Due to geometry net force is zero.

(b) By geometry , $x^2 + \frac{a^2}{4} = a^2$ and $F_1 = F_2$



$$x^2 = \frac{3a^2}{4}$$

$$x = \frac{\sqrt{3}a}{2}$$

$$F_{net} = F = \frac{Gm^2}{x^2} = \frac{4}{3}\frac{Gm^2}{a^2}$$

(c) Initial potential energy = $-\left\{\frac{Gm^2}{a} + \frac{Gm^2}{a} + \frac{Gm^2}{a}\right\}$

$$= -\frac{3Gm^2}{a}$$

Work done on system = Final potential energy – intial potential energy

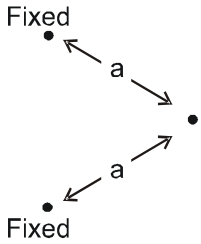
$$= -\frac{3}{2}\frac{Gm^2}{a} - \left\{-\frac{3Gm^2}{a}\right\}$$

$$= \frac{3}{2}\frac{Gm^2}{a}$$ **Ans.**

(d) Initial kinetic energy = 0

Initial potential energy = $-\dfrac{Gm^2}{a} - \dfrac{Gm^2}{a}$

$= -\dfrac{2Gm^2}{a}$



Total initial energy = $-\dfrac{2Gm^2}{a}$

Now, kinetic energy = $\dfrac{1}{2}mv^2$

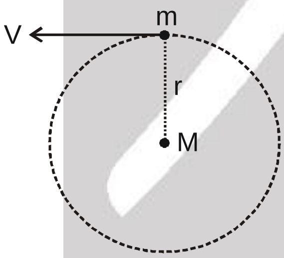
Final Potential Energy = $-\dfrac{Gm^2}{a/2} - \dfrac{Gm^2}{a/2} = -\dfrac{4Gm^2}{a}$

Final Total Energy = $\dfrac{1}{2}mv^2 - \dfrac{4Gm^2}{a}$

$$\frac{2Gm^2}{a} = \frac{1}{2}mv^2$$

$$\sqrt{\frac{4Gm}{a}} = v$$

$$v = 2\sqrt{\frac{Gm}{a}}$$ **Ans.**



D-1.

$$\frac{GMm}{r^2} = \frac{mv^2}{r}$$

$$v = \sqrt{\frac{GM}{r^2}}$$

$$T = \frac{2\pi r}{v} = \frac{2\pi r^{\frac{3}{2}}}{\sqrt{GM}} = \frac{2\pi r^{\frac{3}{2}}}{\sqrt{G\rho \times \frac{4}{3}\pi r^3}}$$

$$T \propto \frac{1}{\sqrt{\rho}}$$ **Ans.**

D-2. Net force on the package is zero hence it will revolve around the earth and never reach to earth surface.

D-3. Kinetic energy decreases with increase in radius while the potential and total energy increases with increase in radius.

D-4. According to kepler's law applying angular momentum conservation $m_1v_1r_1 = m_2v_2r_2$, V_{max} is (a) ans.

D-5. Escape velocity is independent of direction of projection.

E-1. $w_e = 50 \times 10 = 500$ N
$w_p = 50 \times 5 = 250$ N
Hence option A is correct

PART-III

1. P.E. = $-\frac{GMm}{r} \Rightarrow$ K.E. = $\frac{1}{2}mV^2$

Total energy = $-\frac{GMm}{r} + \frac{1}{2}mV^2$

T.E. = 0if $-\frac{GMm}{r} + \frac{1}{2}mV^2 = 0$ $\Rightarrow v = \sqrt{\frac{2GM}{r}}$

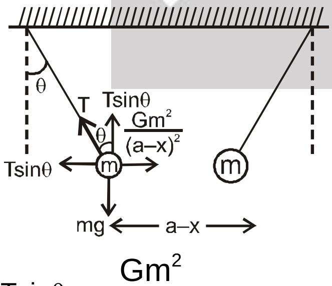
For $v < \sqrt{\frac{2GM}{r}}$, T.E. is – ve

for $v > \sqrt{\frac{2GM}{r}}$, T.E. is + ve

If V is $\sqrt{\frac{GM}{r}}$ i.e. equal to orbital velocity, path is circular.
If T.E. is negative, path is elliptical.
If T.E. is zero, path is parabolic.
If T.E. is positive, path is hyperbolic.

2. (A) At centre of thin spherical shell $V \neq 0$, $E = 0$.
(B) At centre of solid sphere $V \neq 0$, $E = 0$.
(C) At centre of spherical cavity inside solid sphere $V \neq 0$, $E \neq 0$.
(D) At centre of two point masses $V \neq 0$, $E=0$.

EXERCISE-2
PART-I



1.

Tsinθ = $\frac{Gm^2}{(a-x)^2}$

Tcosθ = mg

dividing we get $\tan\theta = \frac{mG}{(a-x)^2 g}$

2. Let's take strip of lenght 'dx' at lenght x, from (0, 0).

Its mass = dm = ρdx = $(a + bx^2)$ dx

Force due to this strip on 'm' = dF = $\dfrac{Gm}{x^2}$ dm = Gm $\dfrac{a+bx^2}{x^2}$ dx

Total force F = $\int dF = \int_{\alpha}^{\alpha+\ell} Gm \dfrac{a+bx^2}{x^2} dx$

$$= Gm \int_{\alpha}^{\alpha+\ell} \left(\frac{a}{x^2}+b\right) dx$$

$$= Gm \left\{\frac{a}{\alpha} - \frac{a}{\alpha+\ell} + b\ell\right\} = Gm\left\{a\left(\frac{1}{\alpha}-\frac{1}{\alpha+\ell}\right)+b\ell\right\}$$

3. Gravitational field at 'm' due to hollowed - out lead sphere

= { Field due to solid spere } – { Field due to mass that was removed }

Field due to solid sphere = $\dfrac{GM}{d^2}$ = E_1 = $\dfrac{GM}{4R^2}$

Field due to removed mass = $\dfrac{GM'}{x^2}$ = E_2

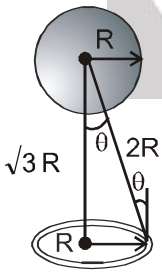
$$M' = \frac{M}{\frac{4}{3}\pi R^3} \times \frac{4}{3}\pi\left(\frac{R}{2}\right)^3 = \frac{M}{8} \quad \text{And} \quad x = d - \frac{R}{2}$$

So, $E_2 = \dfrac{GM}{8\left(d-\frac{R}{2}\right)^2} = \dfrac{GM}{8\left(\frac{3R}{2}\right)^2} = \dfrac{GM}{18R^2}$

$$E_{net} = E_1 - E_2 = \frac{GM}{R^2}\left\{\frac{1}{4}-\frac{1}{18}\right\} = \frac{7GM}{36R^2}$$

$$F_{net} = mE_{net} = \frac{7GMm}{36R^2}$$

4.



$$F = \frac{GMm}{(2R^2)}\cos\theta = \frac{GMm}{(2R^2)} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}GMm}{8R^2}$$

5. $\dfrac{\dfrac{Gm_1}{r_1}}{\dfrac{Gm_2}{r_2}} = \dfrac{3}{4}$

$\dfrac{m_1}{4\pi r_1^2} = \dfrac{m_2}{4\pi r_2^2}$

$m_1 + m_2 = m$

$\dfrac{m}{4\pi R^2} = \dfrac{m}{4\pi R^2}$ or $= \dfrac{Gm}{R \times \dfrac{Gm_1}{r_1}} = \dfrac{5}{3}$ **Ans.**

6. Inside the shell gravitation field due to the shell will be zero but there will be some gravitational field due to the block.

7. From modified Gauss's theorem for gravitation

$\int \vec{E}.d\vec{s} = 4\pi G \left(\int_{r=0}^{r=r} \rho dv \right)$

$E\, 4\pi r^2 = 4\pi G \int_{r=0}^{r=r} \dfrac{k}{r} 4\pi r^2 dr$

get E = constant

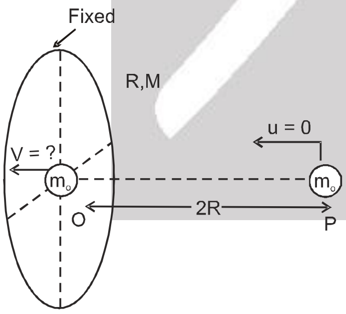
8. as E is constant, so the Potential ($V = -\int E\, dr$) will be proportional to r

9. $\Delta U = m\,(V_f - V_i)$

$\Delta U = m\left(\dfrac{-GM}{(4R)} - \left(\dfrac{-GM}{R} \right) \right) = \dfrac{3}{4} m \left(\dfrac{GM}{R} \right)$

$= \dfrac{3}{4} mR \left(\dfrac{GM}{R^2} \right) = mRg \times 3/4$

10.



Applying energy conservation form P to O

$K_i + U_i = K_f + U_f$

$O + (M_o)\left(\dfrac{-GM}{\sqrt{R^2 + (2R)^2}} \right) = \dfrac{1}{2} m_o V^2 + (m_o)\left(\dfrac{-GM}{R} \right)$

$V = \sqrt{\dfrac{2GM}{R}\left(1 - \dfrac{1}{\sqrt{5}}\right)}$

11. $V_e = \sqrt{\dfrac{2GM}{R}}$

$V = kV_e = K\sqrt{\dfrac{2GM}{R}}$

Initial total energy = $\dfrac{1}{2}mv^2 - \dfrac{GMm}{R}$

$= \dfrac{1}{2}m.k^2\dfrac{2GM}{R} - \dfrac{GMm}{R}$

Final total energy = $\dfrac{1}{2}m0^2 - \dfrac{GMm}{x}$

Applying energy conservation:

$$\frac{1}{2}mk^2.\frac{2GM}{R} - \frac{GMm}{R} = 0 - \frac{GMm}{x}$$

$$\frac{1}{x} = \frac{1}{R} - \frac{k^2}{R} \qquad x = \frac{R}{1-k^2}$$

12. **(C)** For a geo - stationary satellite

$T_{sattelite} = T_{earth}$

$$\frac{4\pi^2}{Gm_e}r^3 = \frac{2\pi}{\omega_{earth}} \; ; \qquad r \propto \frac{1}{\omega_{earth}^{2/3}}$$

As ω_{earth} is doubled So r will be $\dfrac{1}{2^{2/3}} == \dfrac{1}{4^{(1/3)}}$ times

13. Time period of a satellite very close to earth's surface is 84.6 minutes. Time period increases as the distance of the satellite from the surface of earth increase. So, time period of spy satellite orbiting a few hundred km, above the earth's surface should be slightly greater than 84.6 minutes. Therefore, the most appropriate option is (C) or 2 hrs.

14. $\dfrac{mv^2}{R+X} = \dfrac{GM}{(R+X)^2}\times\dfrac{R^2}{R^2}$ or $V = \left[\dfrac{g\;\;R^2}{R+X}\right]^{1/2}$

15. As the magnitude of force does not matter. The torque would still be zero.

16. $g = \dfrac{GM}{R^2} = \dfrac{G(\rho)\left(\dfrac{4}{3}\pi R^3\right)}{R^2} = \dfrac{4}{3}G\pi\rho R$

$g \propto R$

as Radius of the moon is one forth so g on moon is also one fourth.

Time period of a second pendulum on the earth

$$T = 2\pi\sqrt{\frac{\ell}{g_{earth}}}$$

at moon $T = 2\pi\sqrt{\dfrac{\ell}{g_{moon}}}$

dividing $\ell^1 = \ell\dfrac{g_{moon}}{g_{earth}} = \ell\left(\dfrac{1}{4}\right)$

$\ell^1 = \dfrac{99.2}{4} = 24.8$ cm

PART-II

1. $\frac{M_\alpha}{M_\beta} = \frac{5\times10^8}{1\times10^9}$

$\frac{M_\beta}{M_\alpha} = 2$

2. $\left(\frac{GM}{R^2}\right)m_s = m_s\,\omega^2 R$

$M = \frac{\omega^2R^3}{G} = \frac{4\pi^2R^3}{T^2G}$

$M = 6 \times 10^{41}$ kg

$M = nm_s$

$(6 \times 10^{41}) = n\,(2 \times 10^{30})$ $\boxed{n = 3\times10^{11}}$

So x = 11.

Ans. 11

3. $\vec{E} = 3\hat{i} - 4\hat{j}$ N / kg

Then $\vec{F} = m\vec{E} = (1\text{ kg})\,(3\hat{i}-4\hat{j})$N / kg

$= 3\hat{i} - 4\hat{j}$ N

Work done = $W = \vec{F}.\vec{S}$

$\vec{S}$ = Displacement vector

displacement is along $y = \frac{3}{4}x + \frac{9}{4}$

Any point on this line can be written as

$$k_1\hat{i} + \left(\frac{3}{4}k_1 + \frac{9}{4}\right)\hat{j} = k_2\hat{i} + \left(\frac{3}{4}k_2 + \frac{9}{4}\right)\hat{j}$$

$\vec{S}$ = Displacement vector = $(k_2 - k_1)\hat{i} + \frac{3}{4}(k_2 - k_1)\hat{j}$

Now, $W = (3\hat{i} - 4\hat{j}) . \left\{(k_2-k_1)\hat{i} + \frac{3}{4}(k_2-k_1)\hat{j}\right\}$

$= 3(k_2 - k_1) - 3(k_2 - k_1) = 0$

4. The gravitation field is uniform inside the cavity and is directed along $\overrightarrow{OO'}$. Hence the particle will strike at A.

The gravitational field at any point P inside cavity.

$|\vec{E}| = G\rho\,\overrightarrow{PD} - \frac{4\pi}{3}G\rho\,\overrightarrow{PO} = \frac{4\pi}{3}G\gamma\,\overrightarrow{OO'} = \frac{2\pi}{3}G\rho R$

Total workdone = $m\,|\vec{E}|\,.\,\vec{S} = m.\frac{2\pi}{3}G\rho R\,.\,\frac{R}{2}$

Applying work - energy theorem:

Workdone by all force = Change in kinetic energy

$m\,.\frac{2\pi}{3}G\rho R.\frac{R}{2} = \frac{1}{2}mv^2$

$v = \sqrt{\frac{2\pi G\rho R^2}{3}}$ = 2mm/sec.

5. $\Delta PE = \frac{GMm}{R} - \frac{GMm}{2R} = \frac{mgR}{2}$

6. According to conservation of momentum –

$m_R V_R = m_P v_P$

where v_R and v_P are speed of ring and particle in opposite direction, when particle reaches centre of ring.

$2.7 \times 10^9 v_R = 3 \times 10^8 v_P$

$V_P = 9v_R$

By conservation of energy -

$$-\frac{Gm_Rm_P}{\sqrt{R^2+x_0^2}} + 0 = -\frac{Gm_Rm_P}{R} + 1/2\, m_RV_R^2 + 1/2\, m_Pv_p^2$$

$$GM_R \times M_P\left[\frac{1}{R} - \frac{1}{\sqrt{R^2+x_0^2}}\right] = \frac{1}{2} v_P^2 \left[\frac{m_R}{81} + m_P\right]$$

$$6.67 \times 10^{-11} \times 2.7 \times 10^9 \times 3 \times 10^8 \left[\frac{1}{8} - \frac{1}{10}\right] = \frac{1}{2} v_P^2 \left[\frac{2.7\times10^9}{81} + 3\times10^8\right]$$

$v_P = 9$ cm /sec.

7. $W = \frac{3}{5}\frac{GM^2}{R} = \frac{3}{5}\frac{GM}{R^2}MR$

$= \frac{3}{5}g\,MR = \frac{3}{5} \times 10 \times 2.5 \times 10^{31} = 15 \times 10^{31}$ J

8. Speed of particle at A, V_A = escape velocity on the surface of moon $= \sqrt{\frac{2GM}{R}}$

At highest point B, $V_B = 0$

From energy conservation.

$$\frac{1}{2}mV_A^2 = m(V_B - V_A) = m\left(\frac{U_B}{m} - \frac{U_A}{m}\right)$$

or $\frac{V_A^2}{2} = \left(\frac{U_B}{m} - \frac{U_A}{m}\right)$, also $[3R^2 - r^2]$

$$\therefore \frac{GM}{R} = \frac{-GM}{R+h} - \left[\frac{-GM}{R^3}\left[1.5R^2 - 0.5\left[R - \frac{R}{100}\right]^2\right]\right]$$

or $\frac{1}{R} = \frac{-1}{R+h} + \frac{3}{2R} - \frac{1}{2}\left(\frac{99}{100}\right)^2 \frac{1}{R}$

$h = \frac{(100)^2 + (99)^2}{(100)^2 - (99)^2}R$

$h = \frac{19801}{199}R$

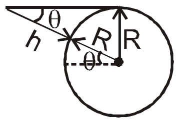
or $h = 99.5\,R \approx$ **99R Ans**

9. $\frac{T_1}{T_2} = \left(\frac{r_1}{r_2}\right)^{3/2}$

or $\frac{T}{5} = (4)^{3/2}$ or T = 40 hr.

10. $\sin\theta = \frac{R}{R+h}$

or θ = colatitude = $\sin^{-1}\left(\frac{R}{R+h}\right)$ = 30°



PART-III

1. In two star problem

$T = 2\pi\sqrt{\frac{r^3}{3Gm}}$

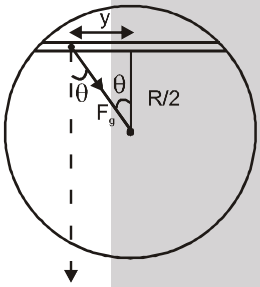
$T \propto r^{3/2}$ and $T \propto \frac{1}{\sqrt{m}}$

2. Inside a uniform spherical shell

$E_{in} = 0$

V_{in} = constant = $\frac{Gm}{R}$

3.*



$F_g \cos\theta$

$F_g = \frac{GMmr}{R^3}$

pressing force = $F_g \cos\theta = \frac{GMmr\cos\theta}{R^3}$

$= \frac{GMm}{2R^2}$ = constant.

$a = \frac{Fg\sin\theta}{m} = \frac{GMr\sin\theta}{R^3}$

$a = \frac{GMy}{R^3}$

4. In case of earth the gravitational field is zero at infinity as well as the the centre and the potential is minimum at the centre .

5. The angular velocity of the geostationary satellite must be equal to angular velocity of earth in both direction and magnitude.

6. For a planetry motion Total mechanical Energy = Constant

Angular momentum about the sun = constant

$\frac{dA}{dt}$ about the sun = constant

7. In elliptical orbit sun is at one of the foci hence the distance between the planet and sun changes as planet revolves hence linear speed, kinetic energy and potential energy of planet donot remain constant

8. $\omega_S = \dfrac{2\pi}{1.5}$, $\omega_E = \dfrac{2\pi}{24}$

$\omega_{\text{west to east}} = 2\pi\left[\dfrac{1}{1.5} - \dfrac{1}{24}\right]$

$T_{\text{west to east}} = \dfrac{2\pi}{\omega_{\text{west to east}}}$ = 1.6 hours

Similarly

$\omega_{\text{east to west}} = 2\pi\left[\dfrac{1}{1.5} + \dfrac{1}{24}\right]$

$T_{\text{east to west}} = \dfrac{24}{17}$ hours

9. **(A)** If it is projected radially it will go up and than move down in a straight line
(B) If it is projected with a small velocity near the earth's surface, g will be almost constant. So its path will be almost parabolic (Projectile Motion).

(C) If the body is projected tangentially with orbital speed $\left(V_o = \sqrt{\dfrac{GM}{r}}\right)$ then it will revolve in circular orbit
(D) If the body is projected with a velocity $V \in (V_o, V_e)$ it may revolve in an elliptical orbit.

10. Escape velocity = $\sqrt{\dfrac{2GM}{R}} = V_e$

Orbital velocity = $\sqrt{\dfrac{GM}{R}} = V_0$

Escape velocity = $\sqrt{2}$ × orbital velocity $\rightarrow$ (A)

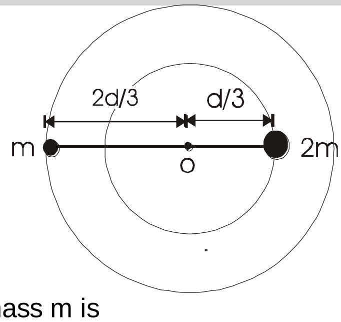
$\dfrac{1}{2}mV_e^2 = 2 \times \dfrac{1}{2} mV_0^2$ $\rightarrow$ (C)

11*. PE = $-G\, m_1 m_2/r$, ME = $-G\, m_1 m_2 / 2r$
On decreasing the radius of orbit PE and ME decreases

12.* the reading of the beam balance will be independent of effective g, so $W_b = W'_b$ but the reading of the spring balance will Proportional to $g_{effactive}$
At equator due to centrifugal force of earth, $g_{effactive}$ is less so Reading of spring ballance is less $W_s < W'_b$

PART-IV

1 to 3 Let the angular speed of revolution of both stars be ω about the common centre, that is, centre of mass of system.



The centripetal force on star of mass m is

$$m\omega^2 \frac{2d}{3} = \frac{Gm(2m)}{d^2}. \text{ Solving we get } T = \sqrt{\frac{4\pi^2}{3Gm} d^3}$$

The ratio of angular momentum is simply the ratio of moment of inertia about center of mass of system.

$$\frac{L_m}{L_M} = \frac{I_m\omega}{I_M\omega} = \frac{m\left(\frac{2d}{3}\right)^2}{2m\left(\frac{d}{3}\right)^2} = 2$$

Similarly, The ratio of kinetic energy is simply the ratio of moment of inertia about center of mass of system.

$$\frac{K_m}{K_M} = \frac{\frac{1}{2}I_m\omega^2}{\frac{1}{2}I_M\omega^2} = \frac{m\left(\frac{2d}{3}\right)^2}{2m\left(\frac{d}{3}\right)^2} = 2$$

4 to 6 $T^2 = \left(\frac{4\pi^2}{GM}\right)R^3$

$R = \left(\frac{GM}{4\pi^2}\right)^{1/3} T^{2/3}$

$\log R = \frac{2}{3}\log T + \frac{1}{3}\log\left(\frac{GM}{4\pi^2}\right)$

$y = mx + c$

(3) Slope $= m = \frac{2}{3}$

intercept $c = \frac{1}{3}\log\left(\frac{GM}{4\pi^2}\right) = 6$

$\log\frac{\left(\frac{20}{3}\times 10^{-11}\right)M}{4\times 10} = 18$

(4) $M = 6 \times 10^{29}$ Kg

(5) $T^2 \propto R^3$

$\left(\frac{R_A}{R_B}\right)^3 = \left(\frac{R_A}{R_B}\right)^2 = \left(\frac{\omega_B}{\omega_A}\right)^2$

$\left(\frac{R}{4R}\right)^3 = \left(\frac{\omega_B}{\omega_A}\right)^2 \Rightarrow \left(\frac{\omega_B}{\omega_A}\right) = \left(\frac{1}{8}\right)$

$\omega_{rel} = 8\omega_0 - \omega_0 = 7\omega_0$

$\theta_{rel} = (\omega_{rel})\, t$

$2\pi = (T\omega_0)\, t$

$t = \frac{2\pi}{T\omega_0}$

7. Let M and R be the mass and radius of the earth respectively. If m be the mass of satellite, then escape velocity from earth $\nu_e = \sqrt{(2Rg)}$

Velocity of satellite $\nu_s = \frac{\nu_e}{2} = \sqrt{(2Rg)}/2$ .... (1)

Further, $\nu_s = \sqrt{\left(\frac{GM}{r}\right)} = \sqrt{\left(\frac{R^2g}{R+h}\right)}$

$\therefore \quad \nu_s^2 = \frac{R^2g}{R+h} \qquad h = R = 6400$ km

8. $T^2 = \dfrac{4\pi^2}{Gm}x^3$

Hence time period of revolution T is

$$T = 2\pi\sqrt{\frac{x^3}{Gm}} \qquad \text{(Put } x = 2R\text{)}$$

$$\therefore \quad T = 2\pi\sqrt{\frac{8R}{g}}$$

9. Now total energy at height h = total energy at earth's surface (from principle of conservation of energy)

$$\therefore \quad 0 - GM\,\frac{m}{R+h} = \frac{1}{2}mv^2 - GM\frac{m}{R}$$

$$\text{or} \quad \frac{1}{2}mv^2 = \frac{GMm}{R} - \frac{GMm}{2R} \qquad (\because h = R)$$

$$\therefore v = \sqrt{gR}$$

EXERCISE-3
PART-I

1. $W_{ext} = U_\infty - U_P$

$$W_{ext} = 0 - \int -\left(-\frac{Gdm}{x}.1\right)$$

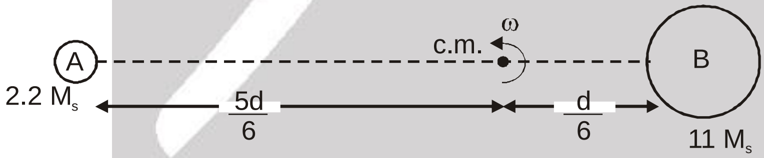
$$W_{ext} = G\int \frac{M}{\pi\times 7R^2}\frac{2\pi rdr}{\sqrt{16R^2+r^2}} = \frac{2GM}{7R^2}\int\frac{rdr}{\sqrt{16R^2+r^2}} = \frac{2GM}{7R^2}\int\frac{zdz}{z} = \frac{2GM}{7R^2}\,[Z]$$

$$W_{ext} = \frac{2GM}{7R^2}\left[\sqrt{16R^2+r^2}\right]_{3R}^{4R}$$

$$W_{ext} = \frac{2GM}{7R^2}\left[4\sqrt{2}R - 5R\right]$$

$$W_{ext} = \frac{2GM}{7R^2}\left[4\sqrt{2} - 5\right].$$

2.



$$\frac{\text{Total angular momentum about c.m.}}{\text{Angular momentum of B about c.m.}} = \frac{(2.2M_s)\left(\omega\frac{5d}{6}\right)\left(\frac{5d}{6}\right) + (11\,M_s)\left(\omega\frac{d}{6}\right)\left(\frac{d}{6}\right)}{(11\,M_s)\left(\omega\frac{d}{6}\right)\left(\frac{d}{6}\right)} = 6.$$

3. $g = \dfrac{GM}{R^2} = \dfrac{(G)\rho\left(\frac{4}{3}\pi R^3\right)}{R^2}$; $g \,\alpha\, \rho R$

$$\frac{g'}{g} = \left(\frac{\rho'}{\rho}\right)\left(\frac{R'}{R}\right) = \left(\frac{2}{3}\right)\left(\frac{R'}{R}\right) = \frac{\sqrt{6}}{11} \quad \text{Given}$$

$$\frac{R'}{R} = \frac{3\sqrt{6}}{22} \qquad V_e = \sqrt{\frac{GM}{R}} = \sqrt{\frac{(G)\left((\rho)\right)\left(\frac{4}{3}\pi R^3\right)}{R}} \quad \Rightarrow \quad V_e \,\alpha\, R\sqrt{\rho}\ ;\ V_e = 3 \text{ km/hr.}$$

4. $V_e = \sqrt{2}v_0$

$KE = \frac{1}{2}mv_e^2 = \frac{1}{2}m\left(\sqrt{2}v_0\right)^2 = mv_0^2$

5*. $V_{es} = \sqrt{\frac{2GM}{R}} = \sqrt{\frac{2.G\rho.\frac{4}{3}\pi R^3}{R}} = \sqrt{\frac{4G\rho}{3}}\ R$

$V_{es} \propto R$

Sarface area of $P = A = 4\pi R_P^2$

Surface area of $Q = 4A = 4\pi R_Q^2$

$\Rightarrow R_Q = 2R_p$

mass R is $M_R = M_P + M_Q$

$\rho\frac{4}{3}\pi R_R^3 = \rho\frac{4}{3}\pi R_P^3 + \rho\frac{4}{3}\pi R_Q^3$

$\Rightarrow R_R^3 = R_P^3 + R_Q^3$

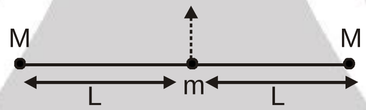
$= 9R_P^3$

$R_R = 9^{1/3} R_P \Rightarrow R_R > R_Q > R_P$

Therefore $V_R > V_Q > V_P$

$\frac{V_R}{V_P} = 9^{1/3}$ and $\frac{V_P}{V_Q} = \frac{1}{2}$

6*. $-\frac{GM.2m}{L} + \frac{1}{2}mv^2 = 0 + 0$



$\Rightarrow v = \sqrt{\frac{4GM}{L}}$

7. Given, $R_{planet} = \frac{R_{earth}}{10}$ and

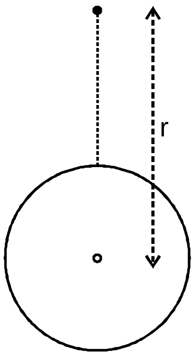
density, $\rho = \frac{M_{earth}}{\frac{4}{3}\pi R_{earth}^3} = \frac{M_{Planet}}{\frac{4}{3}R_{planet}^3}$ $\Rightarrow$ $M_{planet} = \frac{M_{earth}}{10^3}$

$g_{\text{surface of planet}} = \frac{GM_{planet}}{R_{planet}^2} = \frac{GM_e.10^2}{10^3.R_e^2} = \frac{GM_e}{10R_e^2} = \frac{g_{\text{surface of earth}}}{10}$

$g_{\text{depth of planet}} = g_{\text{surface of planet}}\left(\frac{x}{R}\right)$ where x = distance from centre of planet

$T = \int_{4R/5}^{R} \lambda dx\, g\left(\frac{x}{R}\right) = \frac{\lambda g}{R}\left[\frac{x^2}{2}\right]_{4R/5}^{R} = 108\ N$

8. When it reaches its maximum height, its acceleration due to the planet's gravity is 1/4th of its value at the surface of the planet.



$$\frac{GM}{r^2} = \frac{1}{4}\frac{GM}{R^2}$$

r = 2R

By conservation of mechanical energy

$$\frac{-GMm}{R} + \frac{1}{2}mv^2 = \frac{-GMm}{r} + 0$$

$$\Rightarrow \frac{1}{2}mv^2 = \frac{GMm}{2R}$$

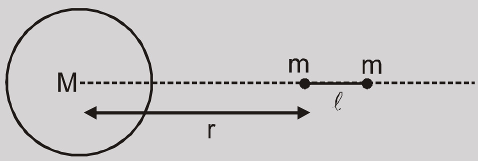
$$v_{esc} = \sqrt{\frac{2GM}{R}} = v\sqrt{N}$$

N = 2

9. For point mass at distance r = 3ℓ

$$\frac{GMm}{(3\ell)^2} - \frac{Gm^2}{\ell^2} = ma$$

For point mass at distance r = 4ℓ



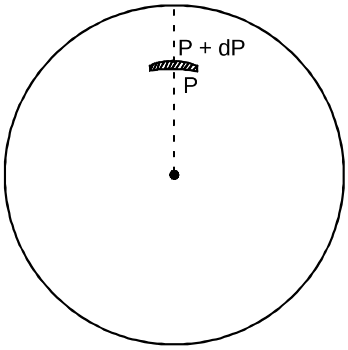
$$\frac{GMm}{(4\ell)^2} + \frac{Gm^2}{\ell^2} = ma$$

$$\frac{GMm}{9\ell^2} - \frac{Gm^2}{\ell^2} = \frac{GMm}{16\ell^2} + \frac{Gm^2}{\ell^2}$$

$$\frac{7GMm}{144} = \frac{2Gm^2}{\ell^2}$$

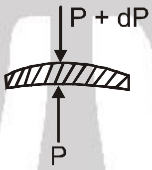
$$m = \frac{7M}{288}$$

10*. Gravitative field at a distance r



$$= \frac{G\rho \frac{4}{3}\pi r^3}{r^2} = \frac{4G\rho\pi r}{3}$$

Consider a small patch of width dr at a distance r



$$-dP.\ \Delta A = \frac{4}{3}\ G\pi\rho r\ \ \Delta A\ dr.\ \rho$$

$$\int_O^P dP = \int_R^r \frac{4G\rho^2\pi}{3} r dr$$

$$-P = \frac{4G\rho^2\pi}{3\times 2}\left[r^2 - R^2\right]$$

$$P = \frac{2G\rho^2\pi}{3}\left[R^2 - r^2\right] \quad \Rightarrow \quad P = c(R^2 - r^2)$$

$$r = \frac{3R}{4} \qquad P_1 = c\left(R^2 - \frac{9R^2}{16}\right) = c\ \frac{7R^2}{16}$$

$$r = \frac{2R}{3} \qquad P_2 = c\left(R^2 - \frac{4R^2}{9}\right) = c\frac{5R^2}{9}$$

$$\frac{P_1}{P_2} = \frac{63}{80}$$

$$r = \frac{3R}{9} \qquad P_3 = c\left(R^2 - \frac{9}{25}R^2\right) = c\frac{16R^2}{25}$$

$$r = \frac{2R}{5} \qquad R_4 = c\left(R^2 - \frac{4R^2}{25}\right) = c\frac{21R^2}{25}$$

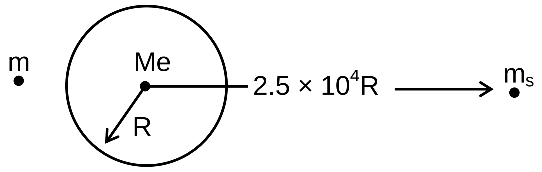
$$\frac{P_3}{P_4} = \frac{16}{21}$$

$$r = R \qquad P_5 = c\left(R^2 - \frac{R^2}{G}\right) = \frac{3R^2}{9}$$

$$r = \frac{R}{3} \qquad P_6 = c\left(R^2 - \frac{R^2}{9}\right) = \frac{8R^2}{9}$$

$$\frac{P_s}{P_r} = \frac{21}{32}$$

11.



Given $\sqrt{\frac{2GM_e}{R}} = 11.2$ km/s

$$\frac{1}{2}mv^2 - \frac{GmM_e}{R} - \frac{GM_sm}{2.5 \times 10^4 R} \geq 0$$

for $v = v_e$

$$v_e^2 = \frac{2GM_e}{R} + \frac{2GM_s}{2.5 \times 10^4 R} = \frac{2GM_e}{R} + \frac{6 \times 10^5 GM_e}{2.5 \times 10^4 R} = \frac{GM_e}{R}(2 + 24) = \sqrt{\frac{26GM_e}{R}} = 40.4 \text{ km/sec.}$$

12. (P) $v_0 = \sqrt{\frac{GM}{R}}$

$$\frac{v_1}{v_2} = \sqrt{\frac{R_2}{R_1}} = \frac{2}{1}$$

(Q) $L = mvR$

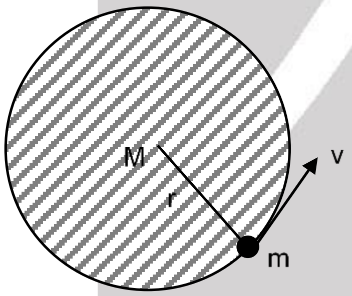
$$\frac{L_1}{L_2} = \frac{m_1}{m_2} = \frac{v_1}{v_2} = \frac{R_1}{R_2} = \frac{2}{1} \text{ x } \frac{2}{1} \text{ x } \frac{1}{4} = 1$$

(R) $KE = \frac{GMm}{R}$

$$\frac{K_1}{K_2} = \frac{m_1}{m_2} \text{ x } \frac{R_2}{R_1} = \frac{2}{1} \text{ x } \frac{4}{1} = 8$$

(S) $\frac{T_1}{T_2} = \left(\frac{R_1}{R_2}\right)^{3/2} = \frac{1}{8}$

13.



$$\frac{GMm}{r^2} = \frac{mv^2}{r}$$

$$= \frac{2}{r}\left(\frac{1}{2}mv^2\right)$$

$$\Rightarrow \quad \frac{GMm}{r^2} = \frac{2K}{r} \quad \Rightarrow \quad M = \frac{2Kr}{Gm}$$

$$\Rightarrow \quad dM = \frac{2K}{Gm}dr \quad \Rightarrow \quad 4\pi r^2 dr\rho = \frac{2K}{Gm}dr \quad \therefore \quad \rho = \frac{K}{2\pi Gmr^2}$$

$$\Rightarrow \quad \frac{\rho}{m} = \frac{K}{2\pi Gm^2r^2}$$

Alternative. $\dfrac{GM(r)}{r^2} = \dfrac{V^2}{r}$ where M(r) = total mass upto radius (r)

$\Rightarrow \quad K = \dfrac{GMm}{2r} \quad \Rightarrow \quad M(r) = \dfrac{2Kr}{Gm}$

$\Rightarrow \quad dM(r) = \dfrac{2K}{Gm}dr = \rho dV = \rho 4\pi r^2 dr$

$\Rightarrow \quad \rho = \dfrac{K}{G2\pi r^2 m} \quad \Rightarrow \quad \dfrac{\rho}{m} = \dfrac{K}{2\pi G m^2 r^2}$

Correct option 4

PART-II

1. $\dfrac{Gm}{x^2} = \dfrac{G(4m)}{(r-x)^2}$

$\dfrac{1}{x} = \dfrac{2}{r-x}$

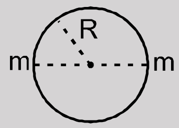
$r - x = 2x$

$3x = \dfrac{r}{3}$

$x = \dfrac{r}{3}$

$-\dfrac{Gm}{r/3} - \dfrac{G(4m)}{2r/3} \quad ; \quad -\dfrac{3Gm}{r} - \dfrac{6Gm}{r} = -\dfrac{9Gm}{r}$

2. $\dfrac{Gm^2}{(2R)^2} = m\omega^2 R$



$\dfrac{Gm^2}{4R^3} = \omega^2$

$\omega = \sqrt{\dfrac{Gm}{4R^3}} \quad \Rightarrow \quad v = \omega R \quad \Rightarrow v = \sqrt{\dfrac{Gm}{4R^3}} \times R = \sqrt{\dfrac{Gm}{4R}}$

3. $W = 0 - \left(-\dfrac{GMm}{R}\right) = \dfrac{GMm}{R}$

$= gR^2 \times m/R = mgR$

$= 1000 \times 10 \times 6400 \times 10^3$

$= 64 \times 10^9$ J$= 6.4 \times 10^{10}$

4. $E_f = \dfrac{1}{2}mv_0^2 - \dfrac{GMm}{3R} = \dfrac{1}{2}m\dfrac{GM}{3R} - \dfrac{GMm}{3R} = \dfrac{GMm}{3R}\left(\dfrac{1}{2} - 1\right) = \dfrac{-GMm}{6R}$

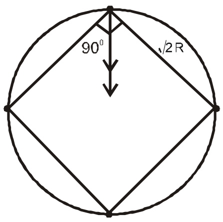
$E_i = \dfrac{-GMm}{R} + K$

$E_i = E_f$

$K = \dfrac{5GMm}{6R}$

Ans (1)

5.



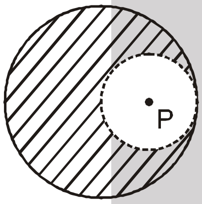
$$2\frac{GM^2}{(\sqrt{2}R)^2}\frac{1}{\sqrt{2}}+\frac{GM^2}{4R^2}=\frac{Mv^2}{R}$$

$$\frac{GM^2}{\sqrt{2}R^2}+\frac{GM^2}{4R^2}=\frac{Mv^2}{R}$$

$$v=\frac{1}{2}\sqrt{\frac{GM}{R}\left[\ 1+2\sqrt{2}\ \right]}$$

6. Potential at point P due to complete solid sphere

$$= -\frac{GM}{2R^3}\left(3R^2-\left(\frac{R}{2}\right)^2\right) = -\frac{GM}{2R^3}\left(3R^2-\frac{R^2}{4}\right)$$



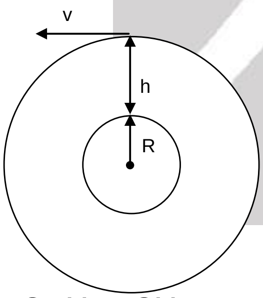
$$= -\frac{GM}{2R^3}\left(\frac{11R^2}{4}\right) = -\frac{11GM}{8R}$$

Potential at point P due to cavity part

$$= -\frac{3}{2}\frac{G\frac{M}{8}}{\frac{R}{2}} = \frac{-3GM}{8R}$$

So potential due to remaining part at point P = $-\frac{11GM}{8R}+\frac{3GM}{8R} = -\frac{GM}{R}$

7.



$$\frac{GmM}{(R+h)^2}=\frac{GMm}{R}\ ;\ V=\sqrt{\frac{GM}{R}}$$

$$\frac{1}{2}mv_1^2-\frac{GMm}{R}=0$$

$$v_1=\sqrt{\frac{2GM}{R}}$$

$$\Delta V=\sqrt{\frac{GM}{R}}(\sqrt{2}-1)=\sqrt{gR}\ (\sqrt{2}-1)$$

8. Let mass of moon be M_m.

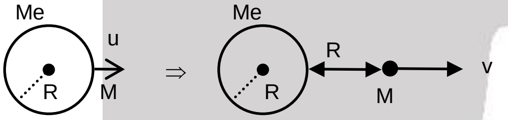
$\therefore$ Mass of Earth $M_e = 64\ M_m$.

$$\frac{M_e}{M_m} = \frac{\frac{4}{3}\pi R_e^3 \times \rho}{\frac{4}{3}\pi R_m^3 \times \rho} = 64 \quad \Rightarrow \frac{R_e^3}{R_m^3} = 64 \Rightarrow R_e = 4R_m$$

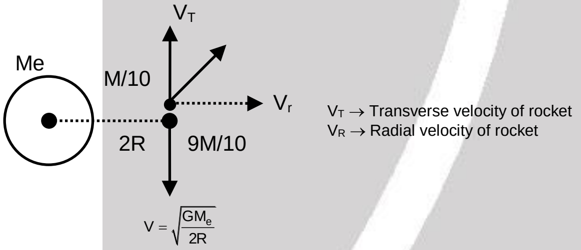
$$\frac{E}{E'} = \frac{\frac{GM_e m}{R_e}}{\frac{GM_m m}{R_m}} = \frac{M_e R_m}{M_m R_e} = \left(\frac{M_e}{M_m}\right) \times \left(\frac{R_m}{R_e}\right) = (64) \times \left(\frac{1}{4}\right) = 16$$

$$\Rightarrow \frac{E}{E'} = 16 \Rightarrow E' = \frac{E}{16}$$

9. $$\frac{-GM_e M}{R} + \frac{1}{2}Mu^2 = \frac{-GM_e M}{2R} + \frac{1}{2}Mv^2$$



$$v = \sqrt{u^2 - \frac{GM_e}{R}}$$



$$\frac{M}{10}V_T = \frac{9M}{10}\sqrt{\frac{GM_e}{2R}}$$

$$\frac{M}{10}V_r = M\sqrt{u^2 - \frac{GM_e}{R}}$$

$$\text{Kinetic energy} = \frac{1}{2}\frac{M}{10}\left(V_T^2 + V_r^2\right) = \frac{M}{20}\left(81\frac{GM_e}{2R} + 100u^2 - 100\frac{GM_e}{R}\right)$$

$$= \frac{M}{20}\left(100u^2 - \frac{119GM_e}{2R}\right)$$

$$= 5M\left(u^2 - \frac{119GM_e}{200R}\right)$$

10. $$3 = \frac{Gm_2}{2^2}$$

$$2 = \frac{Gm_1}{1^2}$$

$$\therefore \frac{3}{2} = \frac{1}{4}\frac{m_2}{m_1}$$

$$\frac{m_1}{m_2} = \frac{1}{6}$$

11. $KE_i + PE_i = KE_f + PE_f$

$$\frac{1}{2}mu_0^2 + \left(-\frac{GMm}{10R}\right) = \frac{1}{2}mv^2 + \left(-\frac{GMm}{R}\right)$$

$$v^2 = u_0^2 + \frac{2GM}{R}\left[1 - \frac{1}{10}\right]$$

$$v = \sqrt{u_0^2 + \frac{9}{5}\frac{GM}{R}} = \sqrt{12^2 + \left(\frac{9}{5}\right)\frac{(11.2)^2}{2}} = \sqrt{144 + 0.9(11.2)^2} = \sqrt{256.896}$$

= 16.028 km/s

$\simeq$ 16

12. Conserving momentum

$$\frac{m}{2}\frac{v}{2} + mv = \left(m + \frac{m}{2}\right)v_f \quad ; \qquad v_f = \frac{5mV}{4 \times \frac{3m}{2}} = \frac{5V}{6}$$

$v_f < v_{orb}$ (= v) thus the combined mass will go on to an elliptical path

HIGH LEVEL PROBLEMS (HLP)

1. The mass of the sun is same for both the cases, so we can apply

$$\left(\frac{T_{planet}}{T_{earth}}\right)^2 = \left(\frac{R_{planet}}{R_{earth}}\right)^3 \quad ; \quad \left(\frac{T_{planet}}{1\,year}\right)^2 = \left(\frac{2r}{r}\right)^3$$

$T_{planet} = 2^{3/2}$ years

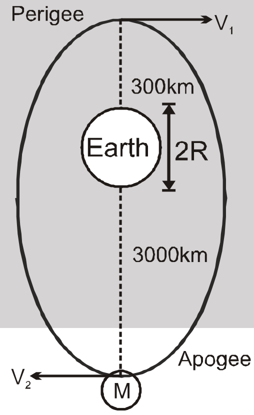
The life span of the man is 70 years . During 70 years , the revolution completed by that planet = $\frac{70}{2^{3/2}} \approx$ 25 revolutions. So he will see 25 summers, 25 winters, 25 springs ... so according to that planet his age will be 25 years.

2. Total distance from apogee to perigee

(a) 300 + 2(6400) + 3000 = 2a

a = 8050 km

Time period of the spacecarft



$$T^2 = \frac{4\pi^2}{GM_e}a^3$$

$$T = \frac{2\pi}{\sqrt{GM_e}}a^{3/2} = \frac{4\pi}{\sqrt{\frac{GM_e}{R^2}R^2}}a^{3/2}$$

$$T = \frac{4\pi}{R\sqrt{g}}a^{3/2} = \frac{4\pi}{6400 \times 10^3 \times \sqrt{9.8}} \times (8050 \times 10^3)^{3/2}$$

(b) Apply angular momentum conservation about the centre of earth, between perigee and Apogee.

$mv_1 r_{min} = m v_2 r_{max}$ ....(i)

$(v_1)(300 + 6400) = v_2(3000 + 6400)$

$$\frac{v_1}{v_2} = \frac{94}{67}$$

(c) Also apply energy conservation between perigee an Apogee

$$\frac{1}{2}mv_1^2 + \left(-\frac{GmM_e}{r_{min}}\right) = \frac{1}{2}mv_2^2 + \left(-\frac{GmM_e}{r_{max}}\right) \quad \text{.............(ii)}$$

Where $r_{min} = (300 + 6400)$km and $r_{max} = (3000 + 6400)$km

From eqn. (i) & (ii) we get

$v_1 = 8.35 \times 10^3$ m/sec

$v_2 = 5.95 \times 10^3$ m/sec.

(d) To escape, velocity at $r \to \infty$ should be zero.

Applying energy conservation between perigee and $r \to \infty$.

$k_i + U = k_f + U_f$

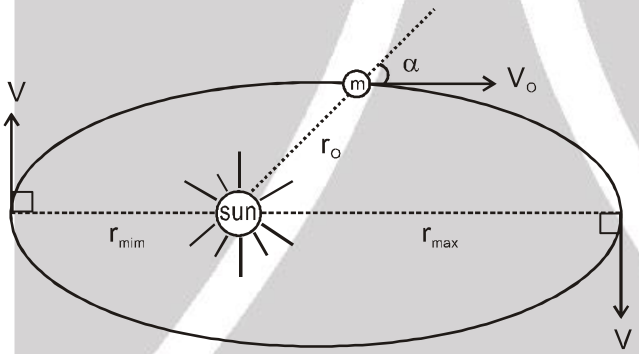
$$\frac{1}{2}mv_1^2 + \left(-\frac{GM_2m}{(300+6400)}\right) = 0 + 0$$

$v_1 = 11.44 \times 10^3$ m/sec.

Increase in speed $= 11.44 \times 10^3 - 8.35 \times 10^3$

$= 3.09 \times 10^3$ m/sec.

3.



at max and min. distance, Velocity will be perpendicular to the radius vector, Applying angular momentum conservation about the sun, between initial position and the position of max or minimum distance.

$MV_0 r_0 \sin\alpha = MVr$ ....(1)

applying energy conservation :

$$\frac{1}{2}MV_0^2 + \left(-\frac{GM_Sm}{r_0}\right) = \frac{1}{2}mv^2 + \left(-\frac{GM_Sm}{r}\right)$$

from equ (1) and (2) get

$$r = \frac{r_o}{2-\eta}\left(1 \pm \sqrt{1-(2-\eta)\eta\sin^2\alpha}\right)$$

where $\eta = \frac{r_o v_o}{GM_s}$

have $+$ sign will give r_{max} and $-$ sign will give r_{min} .

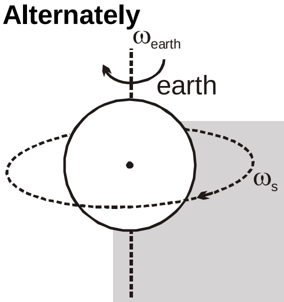
4. R = Radius of earth
r = radius of orbit of geostationary satellite
T = Time period of earth about its axis
$T \propto r^{3/2}$
$\omega \propto r^{-3/2}$

$$\frac{\Delta\omega}{\omega} = \frac{-3}{2} \times \frac{\Delta r}{r} \quad ; \qquad \Delta\omega = \frac{-3}{2} \times \frac{\Delta r}{r} \times \omega$$

$$V_{rel} = (\omega_1 - \omega_2)\, R = -\Delta\omega \times R \quad = \frac{3}{2} \times \frac{\Delta r}{r} \times R \times \omega$$

$$V_{rel} = \frac{3}{2} \times \frac{\Delta r}{r} \times R \times \frac{2\pi}{T} = \frac{3\Delta r R\pi}{rT}$$

Alternately



$$\omega_{earth} = \frac{2\pi}{T_{earth}} = \frac{2\pi}{24\,hours}$$

If the satellite were geo–stationary its T would also be 24 hours. But radius is slightly increased, so its T will also be increased.

$$T^2 = \left(\frac{4\pi^2}{GM}\right) R^3 \text{ ,taking log on both the sides}$$

$$2 \log T = \log \left(\frac{4\pi^2}{GM}\right) + 3\log(R)$$

Differentiating

$$2\frac{dT}{T} = 0 + \frac{3dR}{R}$$

$$dT = \frac{3}{2}\frac{dR}{R}\,T \quad \text{(here R = radius of geo–stationary satllite = 42000 km)}$$

$$dT = \frac{3}{2}\frac{(1 km)}{(42000 km)} \text{ (24 hours)} = \frac{3 \times 24}{2 \times 42000} \text{ hours}$$

Now, angular velocity of satellite relative to the earth

$$\omega_{s/earth} = \omega_s - \omega_{earth} = \frac{2\pi}{T + dT} - \frac{2\pi}{T}$$

$$\omega_{rel} = \frac{2\pi}{T}\left[\left(1 + \frac{dT}{T}\right)^{-1} - 1\right]$$

Using binomial expansion

$$\omega_{rel} = \frac{2\pi}{T}\left(1 - \frac{dT}{T} - 1\right)$$

$$\omega_{rel} = -\frac{2\pi}{T^2} dT$$

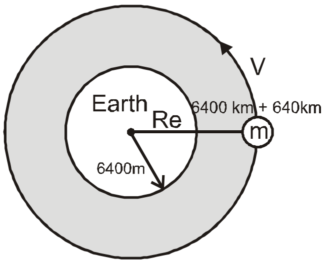
Velocity of the point directly below the satellite relative to earth's surface will be

$$v = (\omega_{rel})R_{earth}$$

$$v = \left(\frac{2\pi}{T^2} dT\right) (R_{earth})$$

$$v = \frac{2\pi}{(24\text{ hours})^2}\left(\frac{3 \times 24}{2 \times 42000}\text{hours}\right) \text{(6400 km).} = \frac{\pi}{189} \text{ m/s} = 1.66 \text{ cm/sec}$$

5.



(a) Orbital speed

$$v = \sqrt{\frac{GM_e}{r}} = \sqrt{\frac{6.6\times10^{-11}(6\times10^{24})}{(6400+640)km}}$$

= 7.53 km/sec.

(b) Time period

$$T^2 = \left(\frac{4\pi^2}{GMe}\right)r^3$$

$$T^2 = \frac{4\pi^2}{(6.6\times10^{-11}\times6\times10^{24})}(6400 + 640\ km)^3$$

T = 1.63 hours

(c) Initial mechanical energy = $-\frac{GM_em}{2r}$

$$= -\frac{(6.6\times10^{-11})(6\times10^{24})(220)}{2\times(6400+640)km}J$$

Total loss in mech. energy during 1500 rev.

$= (1.4 \times 10^5) \times 1500$

$= 21 \times 10^7$ J.

Final mechanical energy remaining after 1500 rev.

$TE_f = TE_i - \Delta E_{loss}$

$$-\frac{GMe_m}{2r_f} - \left(\frac{-(6.6\times10^{-11}\times6\times10^{23}\times220)}{2\times(6400+640)km}\right) - 21\times10^7\ J$$

Solving get r_g = 6812 km

Height from earth's surface = 6812 – 6400 = 412 km

(d) Final orbital velocity

$$v_f = \sqrt{\frac{GM_e}{r_f}} = \sqrt{\frac{6.6\times10^{-11}\times6\times10^{23}}{(6812\ \ km)}}$$

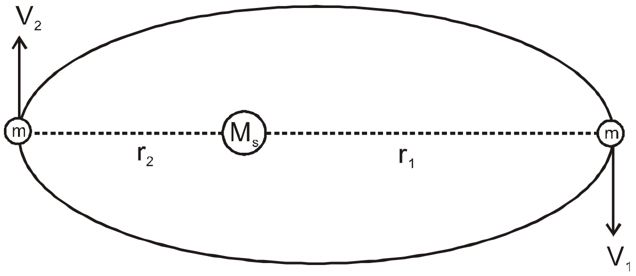
v_f = 7.67 km/sec.

(e) Time period $T = \frac{2\pi r}{v} = \frac{2\times\pi\times6812}{7.67}$ = 1.55 hours

(f) Due to Air resistance, net torque about the earth is non–zero.

So, angular momentum about the earth will not remain conserved.

6.



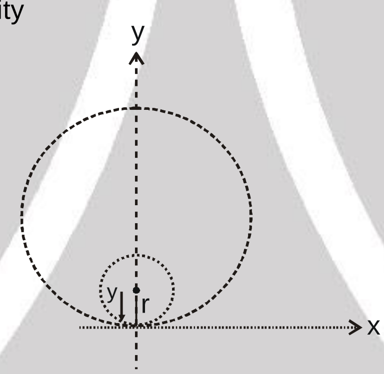
from angular momentum conservation about the sun,
$J = m\, v_1\, r_1 = m\, v_2\, r_2$ ....(1)
from energy conservation

$$\frac{1}{2}\, mv_1^2 + \left(-\frac{GM_S m}{r_1}\right) = \frac{1}{2} mv_2^2 + \left(-\frac{GM_S m}{r_2}\right) \quad ....(2)$$

Solving eq (1) and (2) get

$$J = m\sqrt{\frac{2GM_s r_1 r_2}{(r_1 + r_2)}}$$

7. (a) $r < y < 2r$
Field due to outershell = 0
Distance from centre of solid spere = $(y - r)$
Gravitation field intensity

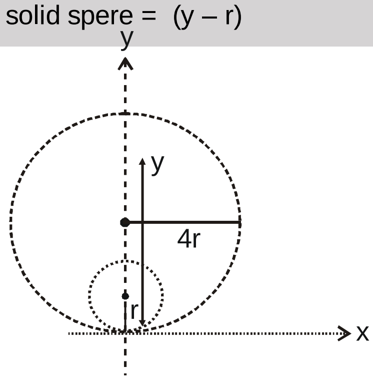


$$= -\frac{GM}{(\text{radius})^3} \times \text{distance from centre}$$

$$= -\frac{GM}{r^3}\,(y - r) \text{ in y - direction}$$

$$= -\frac{GM}{r^3}\,(y - r)\,\hat{j} = \frac{GM}{r^3}\,(y - r)\,(-\hat{j})$$ **Ans.**

(b) Field due to outshell = 0
Distance from centre of solid spere = $(y - r)$



$$E = 0 - \frac{GM}{(y-r)^2}\,\hat{j} = \frac{GM}{(y-r)^2}\,(-\hat{j})$$ **Ans.**

(c) $y > 8r$

For any point outside, the shells acts as point situated at centre.
Distance from centre of hollow shell = $(y - 4r)$

Field due to hollow shell = $-\dfrac{4GM}{(y-4r)}$

Distance from centre of solid spere = $(y - r)$

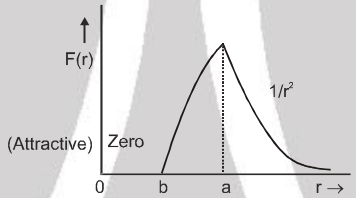
Field due to solid spere = $-\dfrac{GM}{(y-r)^2}$

Total field = $\left\{\dfrac{4GM}{y-4r} + \dfrac{GM}{(y-r)^2}\right\}(-\hat{j})$

8. (a) Force will be due to the mass of the sphere upto the radius r
In case (i) $0 < r < b$; Mass M = 0, therefore F(r) = 0

In case (ii) $b < r < a$; Mass M = $\dfrac{4}{3}\pi\rho\ (r^3 - b^3)$, therefore F(r) = $\dfrac{4}{3}\pi G\rho m\left(r - \dfrac{b^3}{r^2}\right)$

(iii) $a < r < \infty$; Mass $\dfrac{4}{3}\pi\rho$ M = $(a^3 - b^3)$, therefore F(r) = $\dfrac{4}{3}\pi G\rho m\left(\dfrac{a^3 - b^3}{r^2}\right)$

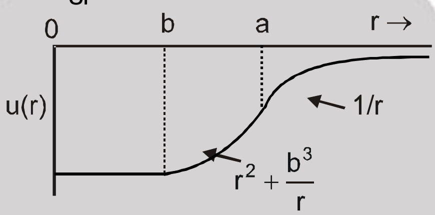


(b) $U_f - U_i = -\int_{r_1}^{r_2} \vec{F}_c . \overrightarrow{dr}$

(i) $0 < r < b$; $u(r) = -2\pi G\rho m(a^2 - b^2)$

(ii) $b < r < a$; $u(r) = \dfrac{-2\pi G\rho m}{3r}(3ra^2 - 2b^3 - r^3)$

(iii) $a < r < \infty$; $u(r) = \dfrac{-4\pi G\rho m}{3r}(a^3 - b^3)$



9. (a) $\dfrac{1}{2}mv^2 = \dfrac{GM_s m}{R}$ or $V = \sqrt{\dfrac{2GM_S}{R}}$

(b) $\dfrac{1}{2}mv_e^2 - \dfrac{GMm}{2R} = 0$ or $V_e = \sqrt{\dfrac{GM}{R}}$

$\dfrac{1}{2}m\,(V + V_e)^2 = \dfrac{GmM_s}{R}$

or $V' + Ve = \sqrt{\dfrac{2GM_s}{R}}$

$V' = \left(\sqrt{2} - 1\right)\sqrt{\dfrac{GM_s}{R}}$

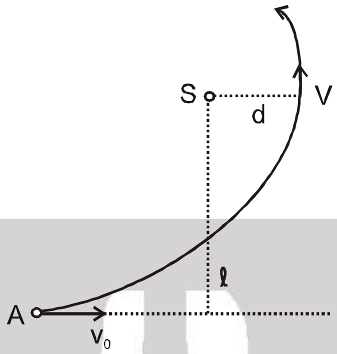
10. Applying angular momentum conservation:

$mv_0\ell = mvd$

$v_0\ell = vd$ ....(i)

Intial energy = $\frac{1}{2}mv_0^2 + 0$

Final energy = $\frac{1}{2}\ mv^2 - \frac{GM_s}{d}$



Applying energy conservation,

$$\frac{1}{2}mv_0^2 = \frac{1}{2}\ mv^2 - \frac{GM_sm}{d}$$

$$v_0^2 = v^2 - \frac{2GM_s}{d} \quad .... \quad \text{(ii)}$$

From equation (i) and (ii) :

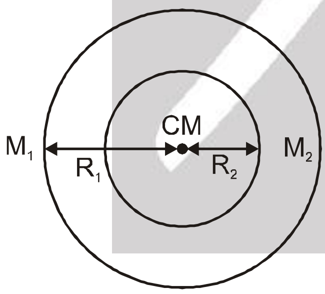
$$v_0^2 = \frac{{v_0}^2\ell^2}{d^2} - \frac{2GM_s}{d}$$

$$d^2 + \frac{2GM_s}{{v_0}^2}d - \ell^2 = 0$$

Solving this quadratic

$$d = -\frac{GM_s}{{v_0}^2} + \sqrt{\left(\frac{GM_s}{{v_0}^2}\right)^2 + \ell^2} = \frac{GM_s}{v_0^2}\left\{\sqrt{1+\left(\frac{\ell v_0^2}{GM}\right)^2} - 1\right\}$$ **Ans.**

11.



(a) Since centre of rotation is the centre of mass of M_1 and M_2.

$M_1 R_1 = M_2 R_2$

or $\frac{M_1}{M_1} = \frac{R_2}{R_1}$ ....(i)

(b) Since force on M_1 and M_2 must be towards CM their radial line should be along same line therefore they must have same orbital period

Now T = $\frac{2\pi R_1}{V_1}$ ....(ii)

and $\frac{GM_1M_2}{(R_1+R_2)^2} = \frac{M_1V_1^2}{R_1}$ ....(iii) from (i), (ii) and (iii)

$$T = 2\pi \frac{(R_1+R_2)^{3/2}}{\sqrt{G\ \ (M_1+M_2)}}$$

(a) Solving equation (i), (ii) and (iii)

We get,

$M_1R_1 = M_2R_2$

$$\frac{M_1}{M_2} = \frac{R_1}{R_2} \qquad \text{......................} \quad \text{(iv)}$$

(b) $T = \dfrac{2\pi R_1}{V_1}$

On solving equation

$$T = \frac{2\pi\,(R_1+R_2)^{\frac{3}{2}}}{\sqrt{G\,(M_1+M_2)}} = \frac{2\pi\, R_2}{V_2} = T_2$$

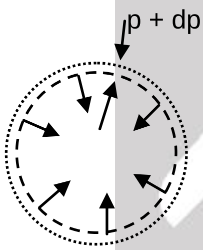
$T_1 = T_2$

Because $\dfrac{R_1}{V_1} = \dfrac{R_2}{V_2}$

12. $dM_r = \rho_r 4\pi r^2 dr$

$$\frac{dM_r}{dr} = \rho_r 4\pi r^2$$

13.



$$\left(\frac{GM_r}{r^2}\right) = \rho_r 4\pi r^2 dr$$

$(P)(4\pi r^2) - (p + dp)(4\pi r^2)$

$$= \frac{GM_r}{r^2} = \Sigma_r 4\pi r^2 dr$$

$$-dp = \frac{GM_r\rho_r}{r^2}\, dr$$

$$\frac{dp}{dr} = -\frac{GM_r}{r^2}\rho_r$$

14. $-\int_{P_c}^{0} dp = \frac{4}{3}\pi G\rho_r^2 \int_0^R r dr$

$P_C = \frac{4}{3}\pi G\rho_r^2 \frac{R^2}{2}$

$P_C = \frac{GM_0^2}{R_0^4} \times \left(\frac{3}{2\pi}\right)$

15. $P_c = \frac{3}{2}\frac{GM_0^2}{\pi R_s^4}$

$= \frac{3 \times 6.67 \times 10^{-11} \times 4 \times 10^{60}}{2 \times 3.14 \times \left(7 \times 10^8\right)^4}$

$= \frac{3 \times 6.67 \times 4 \times 10^{49}}{2 \times 3.14 \times 49 \times 49 \times 10^{32}} \times 10^{17}$

$= \frac{3 \times 4 \times 6.67}{2 \times (3.14) \times (49)^2} \times 10^{17}$

$= 0.00490 \times 10^{17}$

$P_c = 4.9 \times 10^{14} = 5 \times 10^{14}$ N/m^2

16. $PV = nKT$

$P = \frac{2\rho KT}{M_H}$

$4.9 \times 10^{14} = \frac{2 \times 1.40 \times 10^{-23}}{1.67 \times 10^{-27}} \times \rho T$

$4.9 \times 10^{14} = \frac{2.8}{1.67} \times 10^4 \rho T$

$T = 2.10 \times 10^7$