गणितीय औजार (MATHEMATICAL TOOLS)
EXERCISE-1
PART - I

खण्ड (A) :

A-1. $f(\pi/2) = \cos \pi/2 + \sin \pi/2 = 1$

A-2. $f(2) = 4 \times 2 + 3 = 11$

$f\{f(2)\} = f(11) = 4 \times 11 + 3 = 47$

A-3. $\tan 15 = \tan(45-30) = \dfrac{\tan 45 - \tan 30}{1+\tan 45 \tan 30} = \dfrac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}} = \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$

$= \dfrac{(\sqrt{3}-1)^2}{2} = \dfrac{3+1-2\sqrt{3}}{2} = \dfrac{4-2\sqrt{3}}{2} = 2-\sqrt{3}$

A-4. $\cos^2 \theta = 1 - 2\sin^2 \theta$

$2\sin^2 \theta = 1 - \cos^2 \theta \Rightarrow \sin^2 \theta = \left(\dfrac{1-\cos 2\theta}{2}\right)$

A-5. $\sin A \,.\, [\sin A \cos B + \cos A \,.\, \sin B]$

$\sin^2 A \,.\, \cos B + \sin A \,.\, \cos A \,.\, \sin B$

$\sin^2 A \,.\, \cos B + \dfrac{1}{2}\sin 2A \,.\, \sin B$

A-6. $\therefore \cos\left(\dfrac{\pi}{2}+\theta\right) = -\sin\theta$ [cos in II quadrant gives negative value]

$\therefore \cos\left(\dfrac{\pi}{2}+\theta\right) = -\sin\theta$ [II चतुर्थांश में cos, – ve है]

$\sin(\theta - \pi) = \sin[-(\pi - \theta)] = -\sin(\pi - \theta)$

$= -\sin\theta$

$\sin(\pi + \theta) = -\sin\theta$ [sin function is –ve in III quadrant] [III चतुर्थांश में sin फलन –ve है]

A-7. $x_1 = 8 \sin\theta$ and $x_2 = 6\cos\theta$ then $x_1 = 8 \sin\theta$ तथा $x_2 = 6\cos\theta$ तब

$x_1 + x_2 = 10 \sin(\theta + 37°)$

$(x_1 + x_2)_{max} = 10\,[\sin(\theta + 37^o)]_{max} = 10 \times 1 = 10$

खण्ड (B) :

B-1. $\dfrac{dy}{dx} = 2x + 1$

B-2. $\dfrac{dy}{dx} = \sec^2 x - \text{cosec}^2 x$

B-3. $\dfrac{dy}{dx} = \cos x - \sin x$, $\dfrac{d^2y}{dx^2} = -\sin x - \cos x$

B-4. $\dfrac{dy}{dx} = \dfrac{1}{x} + e^x$, $\dfrac{d^2y}{dx^2} = -\dfrac{1}{x^2} + e^x$

खण्ड (C) :

C-1. $\frac{d}{dx}\, e^x \ell nx = \ell n^x \frac{de^x}{dx} + e^x\, \frac{d\ell n^x}{dx}$

$e^x\, \ell nx. + \frac{e^x}{x}$

C-2. $\frac{d(\sin x \cos x)}{dx} = \sin x\, \frac{d(\cos x)}{dx} + \cos x\, \frac{d(\sin x)}{dx} = \cos^2 x - \sin^2 x$

खण्ड (D) :

D-1. $y' = \frac{(3x-2)(2)-(2x+5)(3)}{(3x-2)^2} = \frac{-19}{(3x-2)^2}$

D-2. $\frac{dy}{dx} = \frac{d\left(\frac{\ell nx}{x}\right)}{dx} = \frac{x \times \frac{1}{x} - \ell nx.1}{x^2} = \frac{1-\ell nx}{x^2}$

D-3. $\sec^2 x - \tan^2 x = 1 \qquad \therefore \qquad \frac{d(1)}{dx} = 0$

D-4. (a) $\frac{d}{dx}\, uv = uv^1 + u^1 v$

$= 5 \times 2 + (-1)\,(-3) = 13$

(b) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{vu^1 - uv^1}{v^2} = \frac{(-1)(-3)-(5)(2)}{(-1)^2} = -7$

(c) $\frac{d}{dx}\left(\frac{v}{u}\right) = \frac{uv^1 - vu^1}{u^2} = \frac{(5)(2)-(-1)(-3)}{(5)^2} = \frac{7}{25}$

(d) $\frac{d}{dx}\,(7v - 2u) = 7v^1 - 2u^1$

$= 7 \times 2 - 2\,(-3) = 20$

खण्ड (E) :

E-1. $y = \sin 5x$

माना $5x = \theta$

$y = \sin\theta$

$\frac{dy}{dx} = \frac{dy}{d\theta} \cdot \frac{d\theta}{dx}$

$\frac{dy}{d\theta} = \cos\theta \quad \frac{d\theta}{dx} = 5$

$\therefore\ \frac{d\theta}{dx} = 5\cos\theta \qquad \theta = 5x$

$\therefore\ \frac{dy}{dx} = 5\cos 5x$

E-2. माना $u = (\omega x + \phi)$

$\left(\frac{dy}{dx}\right) = \frac{dy}{du} \times \frac{du}{dx}$

$= 2\cos(\omega x + \phi) \times \omega$

$= 2\omega \cos(\omega x + \phi)$ **Ans.**

E-3. $\frac{dy}{dx} = -27(4-3x)^8$

खण्ड (F) :

F-1. $(x + y)^2 = 4$

$2(x + y)\left(1 + \frac{dy}{dx}\right) = 0$

$\therefore\ x + y \neq 0 \Rightarrow 1 + \frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = -1$

F-2. $x^2y + xy^2 = 6$

$x^2\left[\frac{dy}{dx}\right] + y[+2x] + y^2 + x\left[2y \quad \frac{dy}{dx}\right] = 0$

$\frac{dy}{dx}\ [x^2 + 2xy] + 2xy + y^2 = 0$

$\therefore\ \frac{dy}{dx} = \frac{-2xy - y^2}{x^2 + 2xy}$

खण्ड (G) :

G-1. $\frac{dA}{dt} = \frac{d(\pi r^2)}{dt} = \frac{\pi d(r^2)}{dt} = \frac{2\pi r dr}{dt}$

G-2. $\frac{d}{dt}\ S = \frac{d(4\pi r^2)}{dt} = 8\pi r\frac{dr}{dt}$

खण्ड (H) :

H-1. $x = -t^2 + 4t + 4$ .... (i)

उच्चिष्ठ के लिए

$\frac{dx}{dt} = 0$

$-2t + 4 = 0$

$t = 2$

समीकरण (i) से $\frac{d^2x}{dt^2} = -2 < 0$

t = 2 sec. पर x का मान अधिकतम है। x का अधिकतम मान

$x_{max} = -(2)^2 + 4(2) + 4 = 8$

H-2. $y_{max} = 39$, $y_{min} = 38$

खण्ड (I) :

I-1. $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = 48\ (8x - 1)^2$

I-2. $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$; $3 \cos(3x + 1)$

I-3. $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$; $12x^3$.

I-4. $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$\frac{dy}{dx} = -\frac{1}{3}\sin\frac{x}{3}$

PART - II

खण्ड (A) :

A-1. $y = x^2 - 2x + 1$

$\int y\,dx = \int (x^2 - 2x + 1)\,dx + c = \int x^2 dx \ - 2\int x dx + \int dx \ + c$

$= \frac{x^3}{3} - x^2 + x + c$

A-2. $y = (x)^{1/3} + x^{-1/3}$

$\int y\,dx = \int x^{1/3} dx \ + \int x^{-1/3} dx$

$\frac{3x^{4/3}}{4} + \frac{3x^{2/3}}{2} + c$

A-3. $y = \sec^2 x$ $\qquad \int y\,dx = \tan x + c$

A-4. $y = \text{cosec}^2 x$ $\qquad \int y\,dx = -\cot x + c$

A-5. $y = \sec x \cdot \tan x = \int y\,dx \ \sec x + c$

A-6. $y = \frac{1}{3x}$ $\qquad \int y\,dx = \frac{1}{3}\int \frac{1}{x} dx + c = \frac{1}{3} \ell n x + c$

खण्ड (B) :

B-1. $\int x \sin(2x^2) dx$ माना $u = 2x^2$ $\qquad du = 4x\,dx$

$= \int \sin u \frac{du}{4}$

$= \frac{1}{4}\int \sin u\,du = -\frac{1}{4}\cos u + C$

B-2. $\int \sec 2t \tan 2t\,dt$ $\qquad$ माना $u = 2t$ $\qquad du = 2\,dt$

$= \int \sec u \tan u \frac{du}{2}$

$= \frac{1}{2}\sec u + C$

$= \frac{1}{2}\sec u + C$

B-3. $\int \frac{3}{(2-x)^2}\ dx = \ 3\int (2-x)^{-2}\ dx$

माना $u = 2 - x$ $\qquad du = -\,dx$

$3\int u^{-2}(-dx)$

$= \frac{+3}{u} + C = \frac{3}{2-x} + C$

B-4. $\int \sin(8z - 5)\,dz$

माना Let $u = 8z - 5$ $\qquad du = 8\,dz$

$\int \sin u \frac{du}{8} = \frac{1}{8}\ (-\cos u) + C$

$= -\frac{1}{8}\cos(8z - 5) + C$

खण्ड (C) :

C-1. $\frac{\pi}{2}\int_{-4}^{-1} d\theta = \frac{\pi}{2}[\theta]_{-4}^{-1} = \frac{\pi}{2}$ $[(-1)-(-4)] = \frac{3\pi}{2}$

C-2. $\int_{\sqrt{2}}^{5\sqrt{2}} rdr = \left[\frac{r^2}{2}\right]_{\sqrt{2}}^{5\sqrt{2}} = \frac{(5\sqrt{2})^2}{2} - \frac{(\sqrt{2})^2}{2}$ = 25 – 1 = 24

C-3. $\int_0^1 e^x\,dx = \left[e^x\right]_0^1$ = e – 1

खण्ड (D) :

D-1. y = 2x

क्षेत्रफल = $\int_0^b y\,dx = \int_0^b 2x\,dx = \left[x^2\right]_0^b$ = b^2 units इकाई

D-2. y = $\frac{x}{2}+1$

$\int_0^b y \quad dx = \int_0^b \frac{x}{2}dx + \int_0^b dx = \left[\frac{x^2}{4}\right]_0^b + \left[x\right]_0^b$

= $\frac{b^2}{4}+b$

D-3. $\int_0^{\pi} y\,dx = \int_0^{\pi}\sin x\,dx = \left[-\cos x\right]_0^{\pi}$

= [– cos π + cos 0] = 2

D-4. $\int_0^{\pi}\sin^2 x\,dx = \int_0^{\pi}\frac{1-\cos 2x}{2}$ dx

= $\int_0^{\pi}\frac{1}{2}dx - \int_0^{\pi}\frac{\cos 2x}{2}$ dx

= $\frac{\pi}{2}-\frac{1}{4}\left[\sin 2x\right]_0^{\pi} = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

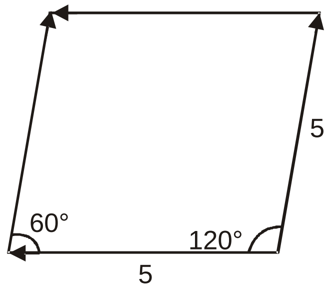
PART - III

खण्ड (A) :

A-1. (i) 105º, (ii) 150º, (iii) 105º.

A-2.

Sol.

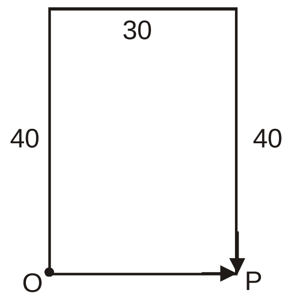


Q = 120°

A-3. $V_R = 5\hat{j}$ m/s = $-5\hat{j}$ m/s.

खण्ड (B) :

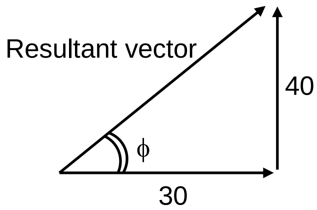
B-1.



$\left|\vec{OP}\right|$ = 30 m

B-2. $\vec{A} = 30\hat{i}$

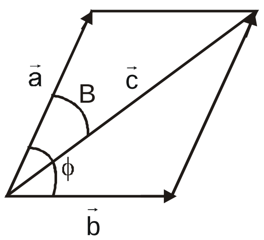
$\vec{B} = 40\ \hat{j}$

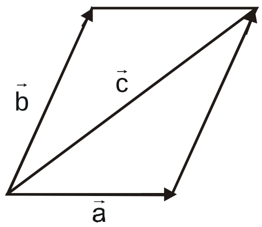


$\hat{i} = \vec{A} + \vec{B} = 30\hat{i} + 40\hat{j}$

$\tan\phi = \dfrac{4}{3} = 53°$

B-3.





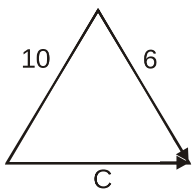
$\left|\vec{C}\right| = \sqrt{a^2 + b^2 + 2ab\cos\theta}$ $\qquad \left|\vec{a}^1\right| = \left|\vec{b}\right|$

$\left|\vec{C}\right| = \sqrt{a^2 + b^2 + 2ab\cos\theta}$ $\qquad \left|\vec{b}^1\right| = \left|\vec{a}\right|$

$\therefore\ \left|\vec{c}^1\right| = \left|\vec{c}\right|$

B-4. शून्य नहीं हो सकता है।

B-5.



त्रिभुज नियम से

C > 10 – 6

C < 10 + 6 $\qquad \therefore\ 4 < c < 16$

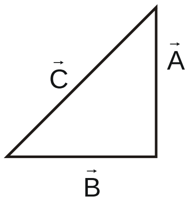
B-6. बन्द बहुभुज बनाने वाले सदिशों का परिणामी शून्य होता है।

B-7. $\left|\vec{A}+\vec{B}\right| = \sqrt{A^2 + \mathbf{B}^2 + 2AB \cdot \cos\theta}$

$\cos\theta$ न्यूनतम होगा जब $\theta = \pi$

B-9. $\vec{C} = \vec{A} + \vec{B}$

$|\vec{A}| = 12 \qquad |\vec{C}| = 13$



$|\vec{B}| = 5$

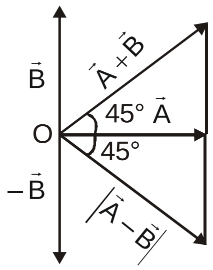
पाइथागॉरिस प्रमेय से $Q_{AB} = 90$

B-10. $\vec{P} + \vec{Q} = \vec{P} - \vec{Q}$

$2\vec{Q} = 0$

$\Rightarrow \vec{Q} = 0$

B-11.



खण्ड (C) :

C-1. $\sqrt{(3)^2+(2)^2+(1)^2} = \sqrt{9+4+1} = \sqrt{14}$ unit.

C-2. x = 25 cos² 10° = – 25 cos 30° [210° = 180° + 30°]

y = 25 sin² 10° = –25 sin 30°

C-3. $\left|\vec{V}\right|$ = 60 km/h

$\left|\vec{V}\right| = \sqrt{30^2 + x^2} = 60$

$30^2 + x^2 = 60°$

$x^2 = 3600 - 900$

$x^2 = 2700$

$x^2 = 900 \times 3$

$x = 30\sqrt{3}$

C-4. $\vec{A} = 0.5\hat{i} + 0.8\hat{j} + c\hat{k}$ इकाई सदिश है

$0.5^2 + 0.8^2 + c^2 = 1^2$

$\therefore \qquad c = \pm \frac{\sqrt{11}}{10}$

C-5. $\vec{A} = 2\hat{i} + 2\hat{j}$ $\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}|\,|\vec{B}|} = \frac{2+2\sqrt{3}}{(2\sqrt{2})(2)}$

$\vec{B} = \hat{i} + \sqrt{3}\,\hat{j}$

$= \frac{2(1+\sqrt{3})}{4(\sqrt{2})} = \frac{1+\sqrt{3}}{2\sqrt{2}} \Rightarrow Q = 15°$

C-6. $\vec{F} = 2\hat{i} - 3\hat{j}$ N

C-7. A(1,1,–1) B(2, –3 , 4)

$\overrightarrow{AB} = \vec{B} - \vec{A}$ = (2, –3, 4) – (1, – 1, 1) = (1, –4, 5)

$\therefore\ \overrightarrow{AB} = \hat{i} - 4\hat{j} + 5\hat{k}$

खण्ड (D) :

D-1. $\vec{A} = \hat{i} + \hat{j} + \hat{k}$ $\vec{B} = 2\hat{i} + \hat{j}$

(a) $\vec{A}\cdot\vec{B}$ = (1)(2) + (1)(1) + (1)(0) = 2 + 1 = 3

$$\vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{vmatrix}$$

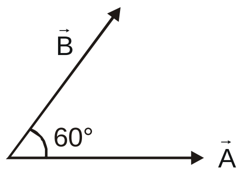
$= \hat{i}(0-1) - \hat{j}(0-2) + \hat{k}(1-2)$

$\vec{A}\times\vec{B} = -\hat{i} + 2\hat{j} - \hat{k}$

D-2. $|\vec{A}| = 4\ |\vec{B}| = 3\quad \theta = 60°$

$\vec{A}\cdot\vec{B} = |\vec{A}|\,|\vec{B}|\cos 60°$

= 4 × 3 × 1/2 = 6



$|\vec{A}\times\vec{B}| = |\vec{A}|\,|\vec{B}|\sin\theta = 4\times 3\times\frac{\sqrt{3}}{2} = 6\sqrt{3}$

D-3. $\vec{A}$, $\vec{B}$ तथा अशून्य सदिश है

$\vec{A}\cdot\vec{B} = 0$ & $\vec{A}\cdot\vec{C} = 0$

$\vec{A}$, $\vec{B}$ के लम्बवत् है।

$\vec{A}$, $\vec{C}$ के लम्बवत् है।

$\therefore\ \vec{B}\times\vec{C}$, $\vec{A}$ के समान्तर है।

D-4. $\vec{A}\cdot\vec{B} = 8$ A B cos θ = 8

$|\vec{A}\times\vec{B}| = 8\sqrt{3}$ AB $\sin\theta = 8\sqrt{3}$

$\tan\theta = \pm\sqrt{3}$ θ = 120°

EXERCISE-2
PART - I

1. $f(x) = \frac{x-1}{x+1}$ .... (i)

$f\{f(x)\} = \frac{f(x)-1}{f(x)+1} = \frac{\frac{x-1}{x+1}-1}{\frac{x-1}{x+1}+1} = \frac{\frac{-2}{x+1}}{\frac{2x}{x+1}} = \frac{-1}{x}$

2. $f(y) = f[f(x)] = \frac{2\left[\frac{2x-3}{3x-2}\right]-3}{3\left[\frac{2x-3}{3x-2}\right]-2} = x$

3. $120° + \theta + \theta = 180°$

$\Rightarrow \quad \theta = 30°$

$\frac{\sin 120}{10} = \frac{\sin\angle A}{a} = \frac{\sin\angle C}{c}$

भुजा a = भुजा c = $\frac{10\sqrt{3}}{3}$m

5. (A) $\frac{3}{5}+\frac{4}{5}=\frac{4}{5}+\frac{3}{5}$ (B) $\frac{3}{5}-\frac{4}{5}=\frac{3}{5}-\frac{4}{5}$

(C) $\frac{3}{4}+1\neq\frac{4}{3}-1$ (D) $\frac{3}{4}\times\frac{4}{3}=1$

6. $R^2 = 2A^2(1+\cos\theta) = 2A^2\left(1+2\cos^2\frac{\theta}{2}-1\right) = 2^2A^2\cos^2\frac{\theta}{2}$

$R = 2A\cos\frac{\theta}{2}$

7. $y = \ln x^2 + \sin x$

$\frac{dy}{dx} = \frac{d(\ln x^2)}{dx} + \frac{d(\sin x)}{dx}$

$= \frac{1}{x^2}\frac{d}{dx}(x^2) + \cos x$

$= \frac{1}{x^2} . 2x + \cos x$

$= \frac{2}{x} + \cos x$

$\Rightarrow \quad \frac{d^2y}{dx^2} = \frac{d(\frac{2}{x})}{dx} + \frac{d(\cos x)}{dx}$

$= \frac{-2}{x^2} - \sin x$

8. $\frac{dy}{dx}=\frac{d(x^{1/7})}{dx}+\frac{d(\tan x)}{dx}$ = $\frac{x^{-\frac{6}{7}}}{7}$ + $\sec^2 x$

$\frac{d^2y}{dx^2}=\frac{1}{7}\frac{d(x^{-6/7})}{dx}+\frac{d(\sec^2 x)}{dx}$ = $\frac{-6}{49}x^{-13/7}$ + 2 secx (secx tan x)

= $\frac{-6}{49}x^{-13/7}$ + 2 tanx $\sec^2$x

9. y = $\left(x+\frac{1}{x}\right)\left(x+\frac{1}{x}+1\right)$

$\frac{dy}{dx}=\left(x+\frac{1}{x}\right)\frac{d(x-\frac{1}{x}+1)}{dx}+\left(x-\frac{1}{x}+1\right)\frac{d\left(x+\frac{1}{x}\right)}{dx}$

$\left(x+\frac{1}{x}\right)$ + $\left[\frac{dx}{dx}-\frac{d\ (\frac{1}{x})}{dx}+\frac{d\ (1)}{dx}\right]+\left(x-\frac{1}{x}+1\right)\left[\frac{dx}{dx}+\frac{d(\frac{1}{x})}{dx}\right]$

= $\left(x+\frac{1}{x}\right)\left(1+\frac{1}{x^2}\right)$ + $\left(x-\frac{1}{x}+1\right)\left(1-\frac{1}{x^2}\right)$

= x + $\frac{1}{x}$ + $\frac{1}{x}$ + $\frac{1}{x^3}$ + x – $\frac{1}{x}$ – $\frac{1}{x}$ + $\frac{1}{x^3}$ + 1 – $\frac{1}{x^2}$

= $\frac{2}{x^3}$ + 2x – $\frac{1}{x^2}$ + 1

10. r = (1 + secθ) sinθ

r = sinθ + secθ sinθ

r = sinθ + tanθ

$\frac{dr}{d\theta}$ = cos θ + $\sec^2\theta$

11. $\frac{dy}{dx}=\frac{(1+\cot x)\frac{d}{dx}(\cot x)-\cot x\frac{d}{dx}(1+\cot x)}{(1+\cot x)^2}$

= $\frac{(1+\cot x)(-\operatorname{cosec}^2 x)-\cot x(-\operatorname{cosec}^2 x)}{(1+\cot x)^2}$ = $\frac{-\operatorname{cosec}^2 x}{(1+\cot x)^2}$

12. $\frac{dy}{dx}=\frac{\tan x\frac{d}{dx}(\ell nx+e^x)-(\ell nx+e^x)\frac{d}{dx}\tan x}{(\tan x)^2}$

= $\frac{\tan x\left(\frac{1}{x}+e^x\right)-\left(\ell nx+e^x\right)(\sec^2 x)}{(\tan x)^2}$

13. $\frac{d}{dx}$ ($\sin^3$x) + $\frac{d}{dx}$ (sin 3 x)

= $\frac{d\sin^2 x}{d\sin x}\times\frac{d\sin x}{dx}+\frac{d\sin 3x}{d(3x)}\times\frac{d(3x)}{dx}$ = 3 $\sin^2$x cos x + $\frac{\cos 3x}{3}$

= 3$\sin^2$x cosx + 3 cos3x

14. $\frac{dy}{dx} = \frac{d}{dx}$ [$\sin^2 (x^2 + 1)$]

$= \frac{d\sin^2(x^2+1)}{d\sin(x^2+1)} \times \frac{d\sin(x^2+1)}{d(x^2+1)} \times \frac{d(x^2+1)}{dx}$

$= 4x \sin(x^2 + 1) \cos(x^2 + 1)$

15. $\frac{dq}{dr} = \frac{d}{dr} (2r - r^2)^{1/2}$

$= \frac{d(2r-r^2)^{1/2}}{d(2r-r^2)} \times \frac{d(2r-r^2)}{dr}$

$= \frac{1}{2}(2r - e^2)^{-1/2}(2 - 2r)$

$= \frac{1-r}{\sqrt{2r-r^2}}$

16. $x^3 + y^3 = 18xy$

$\frac{d(x^3+y^3)}{dx} = \frac{d(18xy)}{dx}$

$\frac{dx^3}{dx} + \frac{dy^3}{dx} = 18x.\frac{dy}{dx} + y.\frac{d(18x)}{dx}$

$3x^2 + 3y^2.\frac{dy}{dx} = 18x.\frac{dy}{dx} + y.18$

$3x^2 + 3y^2.\frac{dy}{dx} = 18x.\frac{dy}{dx} + 18y$

$3y^2 \frac{dy}{dx} - 18x.\frac{dy}{dx} = 18y - 3x^2$

$\frac{dy}{dx}(3y^2 - 18x) = 18y - 3x^2$

$\frac{dy}{dx} = \frac{(18y-3x^2)}{(3y^2-18x)}$

17. $V = \pi r^2 h.$

$\frac{dv}{dt} = \frac{d\pi r^2 h}{dt} = \pi\frac{dr^2 h}{dt}$

$= \pi\left\{h\frac{dr^2}{dt} + r^2\frac{dh}{dt}\right\}$

$= \pi\left\{2rh\frac{dr}{dt} + r^2\frac{dh}{dt}\right\}$

$= \pi\{2rh \cdot 5 + r^2.5\}\left\{\frac{dh}{dt} = 5m/s \ \& \ \frac{dr}{dt} = 5m/s\right\} = \pi\{10rh + 5r^2\}$

18. $x + y = 60$

$x = 60 - y$

$xy = (60 - y)y$

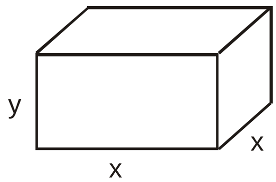
$f(y) = (60 - y)y$

अधिकतम के लिए $f'(y) = 60 - 2y = 0$

$y = 30$

अतः $x = 30$ & $y = 30$

19. माना टंकी के आधार की लम्बाई x व ऊँचाई y है तो क्षेत्रफल = $x^2 + 4xy$.
फिर से x तथा y पृष्ठीय क्षेत्रफल 40 m^2 से इस प्रकार सम्बंधित हैं कि



$x^2 + 4xy = 40$

$y = \frac{40 - x^2}{4x}$

आयतन $v = x^2\left(\frac{40 - x^2}{4x}\right) = \frac{40x - x^3}{4}$

अधिकतम आयतन के लिए

$v'(x) = \frac{40 - 3x^2}{4} = 0$

$x = \sqrt{\frac{40}{3}}$ and $v''(x) = -\frac{3x}{2}$

$v''\sqrt{\frac{40}{3}} = -\frac{3}{2}\sqrt{\frac{40}{3}} < 0$

अतः आयतन $x = \sqrt{\frac{40}{3}}$ m पर अधिकतम हो

PART - II

1. $= \int (x^{-2} + x^{-3})\,dx$

$= \int x^{-2}dx + \int x^{-3}\,dx$

$= \frac{x^{-2+1}}{-\;2+1} + \frac{x^{-3+1}}{-3+1} + C$

$= -x^{-1} - \frac{1}{2}x^{-2} + C,$

2. $\int (1 - \cot^2 x)dx = \int 1 - (\operatorname{cosec}^2 x - 1)dx$

$= \int (2 - \operatorname{cosec}^2 x)dx$

$= \int 2dx - \int \operatorname{cosec}^2 x\,dx$

$= 2x + \cot x + C$

3. $\int \cos\theta\,(\tan\theta + \sec\theta)\,d\theta$

$= \int \cos\theta\tan\theta\,d\theta + \int$ osθ secθ dθ

$= \int \cos\theta\frac{\sin\theta}{\cos\theta}d\theta + \int d\theta$

$= -\cos\theta + \theta + C$

4. $\int 1\,2\,(y^4 + 4y^2 + 1)^2\;(y^3 + 2y)\,dy$

प्रतिस्थापन द्वारा $u = y^4 + 4y^2 + 1$

$du = dy = \frac{d\left(y^4 + 4y^2 + 1\right)}{dy}$ $(4y^3 + 8y)\, dy = 4\,(y^3 + 2y)\, dy$

अतः $\int 12\ (y^4 + 4y^2 + 1)^2 (y^3 + 2y)\, dy = \int\ 3u^2 du$

$= 3\int\ u^2 du = \frac{3u^3}{3} + c = (y^4 + 4y^2 + 1)^3 + C$

5. $\int \frac{dx}{\sqrt{5x+8}}$.

प्रतिस्थापन द्वारा $5x + 8 = u$,

$\frac{d(5x+8)}{dx} = \frac{du}{dx}$

$5 = \frac{du}{dx}$

$5dx = \left(\frac{du}{dx}\right) dx$

$dx = \frac{du}{5}$ तब, $\int \frac{du}{5\sqrt{u}} = 1/5 \int u^{-\frac{1}{2}}\, du = \frac{1}{5} \frac{u^{\ 1/2}}{1/2} + C$

$= \frac{2}{5}\sqrt{u} + C = \frac{2}{5}\ \sqrt{5x+8} + C$

6. माना $u = 3 - 2s \Rightarrow du = -2\, ds$

$\int \sqrt{u}\ \left(\frac{-du}{2}\right) = \frac{-1}{2 \times 3/2} u^{3/2} + C$

$= \frac{-1}{3}\ (3 - 2s)^{3/2} + C$

7. $\int \sec^2(3x+2)dx$

माना $u = 3x + 2$ $\qquad du = 3dx$

$= \int \sec^2 u \frac{du}{3}$

$= \frac{1}{3} \tan u + C = \frac{1}{3} \tan (3x + 2) + C$

8. $\int \csc\left(\frac{\upsilon - \pi}{2}\right) \cot\left(\frac{\upsilon - \pi}{2}\right)\ d\upsilon$

माना $u = \frac{\upsilon - \pi}{2}$

$du = \frac{dv}{2}$

$= \int \csc u \cot u (2du)$

$= -2 \csc u + C$

$= -2 \csc \left(\frac{\upsilon - \pi}{2}\right) + C$

9. $\int \frac{6\cos t}{(2+\sin t)^3}$,

प्रतिस्थापित करते हुए

$2 + \sin t = u$

$\cos t = \frac{du}{dt}$

$\cos t \; dt = du$

अतः $\int \frac{6du}{u^3} = 6\int \frac{du}{u^3}$

$= \frac{6u^{-3+1}}{-3+1} + C = -3(2 + \sin t)^{-2} + C$

10. $\left[\frac{\theta^2}{2}\right]_{\pi}^{2\pi} = \frac{4\pi^2}{2} - \frac{\pi^2}{2} = \frac{3\pi^2}{2}$ **Ans.**

11. $= \left[\frac{x^3}{3}\right]_0^{\sqrt[3]{7}}$

$= \frac{\left((7)^{1/3}\right)^3}{3} - 0 = \frac{7}{3}$

12. प्रतिस्थापित करते हुए

$x^2 = u$,

या $2x\,dx = du$

$x\;dx = \frac{du}{2}$.

अब, सीमायें बदलने पर

$x = 0, \quad u = 0$

$x = \sqrt{\pi}, \quad u = 2$

अत : $\int_{u=0}^{u=\pi} \sin u\, du = [-\cos u]_0^{\pi}$

$= -\cos\pi + \cos 0 = 2$

13. $\left[\frac{1}{3}\ln(3x+2)\right]_0^1$

$= \frac{1}{3}(\ell n\, 5 - \ell n 2)$

$= \frac{1}{3}\ell n\, \frac{5}{2}$.

$= \ell n \left(\frac{5}{2}\right)^{1/3}$

14. दिये गए वक्र तथा x अक्ष के मध्य क्षेत्र का अन्तराल [0, b] पर क्षेत्रफल

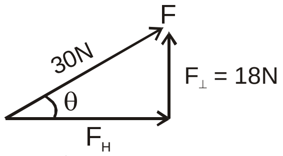
$= \int_0^b 3x^2 dx$

$= \left[\frac{3x^3}{3}\right]_0^b = \left[x^3\right]_0^b = b^3 - 0 = b^3$

PART - III

1. $\vec{B} = \lambda\vec{A} = -4 \times 3$ N-E
= 12 S-W
नहीं यह समान भौतिक राशि को प्रदर्शित नहीं करेगा।

2.



दिया है $F_\perp = 18$

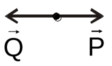
चित्र से $F_H = \sqrt{F^2 - F_1^2} = \sqrt{30^2 - 18^2}$

$= \sqrt{576} = 24N$

चित्र से $\tan\theta = \frac{F_1}{F_H} = \frac{18}{24}$

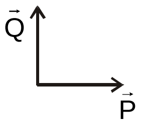
$\tan\theta = 3/4 \quad \Rightarrow \quad \theta = 37°$

3. माना $\vec{P}$ & $\vec{Q}$ दो सदिश हैं
यदि



P – Q = 10 इकाई .... (1)

यदि



$\sqrt{P^2 + Q^2} = 50$ इकाई
$P^2 + Q^2 = 50^2$ .... (2)
समीकरण (1) से
$(10 + Q)^2 + Q^2 = 50^2$
$2Q^2 + 20Q + 100 = 2500$
$2Q^2 + 20Q = 2400$
$Q^2 + 10Q - 1200 = 0$
$(Q + 40)\ (Q - 30) = 0$
Q = 30
अतः समीकरण (1) से
P = 10 + Q
= 10 + 30 = 40 इकाई

4. $\tan\alpha = \frac{\beta\sin\theta}{A + B\cos\theta}$

$\tan\alpha = \frac{6\sin 90}{8 + 6\cos 90} = \frac{1}{8} = \frac{3}{4}$

$\tan\alpha = 3/4$
$\alpha = 37$ Ans.

5. $\overrightarrow{OA} = r\hat{j}$

$\overrightarrow{OC} = r\hat{i}$

$\overrightarrow{OB} = r\,(\cos 45\,\hat{i} + \sin 45\,\hat{j})$

$\overrightarrow{OB} = \frac{r}{\sqrt{2}}\hat{i} + \frac{r}{\sqrt{2}}\hat{j}$

$\vec{R} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}$

$= r\hat{j} + \frac{r}{\sqrt{2}}\hat{i} + \frac{r}{\sqrt{2}}\hat{j} + r\hat{i}$

$|\vec{R}| = \left|\left(r + \frac{r}{\sqrt{2}}\right)\hat{i} + \left(r + \frac{r}{\sqrt{2}}\right)\hat{j}\right| = r\left(1 + \sqrt{2}\right)$

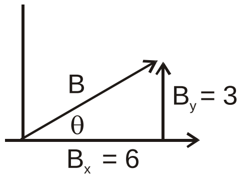
7. दिया है

$\vec{A} = 4\hat{i} + 6\hat{j}$ .... (1)

$\vec{A} + \vec{B} = 10\hat{i} + 9\hat{j}$ .... (2)

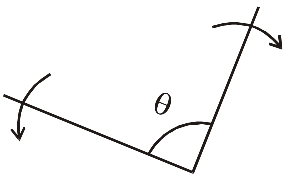
from equation (2) समीकरण (2) से

$\vec{B} = 10\hat{i} + 9\hat{j} - \vec{A} = 10\hat{i} + 9\hat{j} - (4\hat{i} + 6\hat{j}) = 6\hat{i} + 3\hat{j}$



चित्र से $\tan\theta = \frac{B_y}{B_x} = \frac{3}{6} = \tan^{-1}\left(\frac{1}{2}\right)$

9. दो सदिशों के मध्य कोण 180° से अधिक कभी नहीं हो सकता।



θ को बढ़ाने पर, परिणामी सदिश का परिमाण घटता है।

10. प्रारम्भिक वेग = $50\hat{j}$

अंतिम वेग = $-50\hat{j}$

1 परिवर्तन = $50\sqrt{2}$

दक्षिण–पश्चिम दिशा के अनुदिश

11. किन्हीं भी 3 सदिशों का योग चौथे से अधिक होना चाहिये।

12. $a + b > |\vec{a} + \vec{b}| > a - b$

13. $|\vec{A} + \vec{B}| = |\vec{A}| = |\vec{B}|$

Case -I या तो $|\vec{A}| = |\vec{B}| = 0$ (शून्य सदिश)

Case - II $|A| = |B| \neq 0$

$|\vec{A} + \vec{B}| = A^2 + B^2 + AB\cos\theta$

$|A| = |B|$

$= 2A^2 + 2A^2\cos\theta$

$= 2A^2(1 + \cos\theta)$

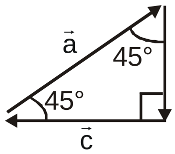
$= 2A\left(2\cos^2\frac{\theta}{2}\right)$

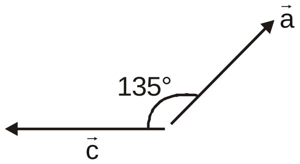
$= 4A\cos^2\frac{\theta}{2}$

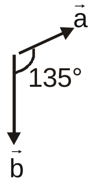
अब $|A + B| = A$

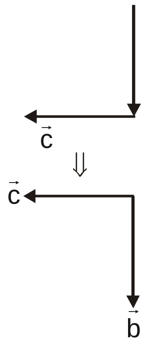
$\cos\left|\frac{\theta}{2}\right| = \frac{1}{2}$ $\qquad \frac{\theta}{2} = 60°$

$\theta = 120°$

14.  $\vec{b}$ सदिश से

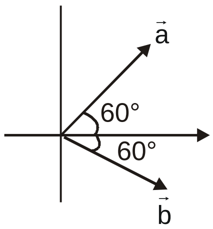






$\therefore$ 90º, 135º, 135º

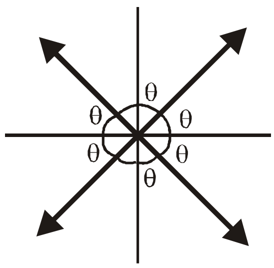
15.



केवल क्षैतिज में + x-अक्ष के अनुदिश

$\Rightarrow 2\cos 60° + 2\cos 60° = 2$

16.



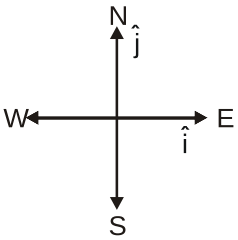
तल (xy तल) में सभी समतलीय हैं।

$6\theta = 360°$

$\theta = 60°$

प्रत्येक के घटक एक दूसरे को निरस्त कर देते हैं अतः परिणामी = 0

17.



$\vec{A} \rightarrow -\hat{k}$

$\vec{B} \rightarrow +\hat{i}$

$\vec{A} \times \vec{B} = -\hat{k} \times \hat{i} = -\hat{j} \Rightarrow$ south दक्षिण

18. किसी भी सदिश को उसके घटक सदिशों के योग के रूप में प्रदर्शित कर सकते हैं।

अर्थात् $\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}$

किसी भी सदिश को उसके घटक सदिशों के योग के रूप में प्रदर्शित कर सकते हैं। अर्थात्

19. $a^2 + b^2 + 2ab\cos\theta = a^2 + 4b^2 - 4ab\cos\theta$

या $\cos\theta = \frac{b}{2a} < 1$

$\therefore$ $b < 2a$

EXERCISE-3

भाग - I

1. (A) $\int \sec x \tan x\, dx = \sec x + C$

(B) $\int \operatorname{cosec} kx \cot kx\, dx = \frac{-\operatorname{cosec} kx}{k} + C$

(C) $\int \operatorname{cosec}^2 kx\, dx = -\frac{\cot kx}{k} + C$

(D) $\int \cos kx\, dx = \frac{\sin kx}{k} + C$

2. (A) $|\vec{A}+\vec{B}| = A^2 + B^2 + 2AB\cos\theta$

$A^2 = A^2 + A^2 + 2A^2\cos\theta$ $\cos\theta = -\frac{1}{2}$, $\theta = 120$

(B) $F_1 \sim F_2 \le R \le F_1 + F_2$

यहाँ $F_1 \sim F_2 = 4$ तथा $F_1 + F_2 = 12$

(C) $\cos\theta = \frac{\vec{A}\ .\ \vec{B}}{|A||B|} = \frac{0}{2\sqrt{2}\times 3} = 0$ $\theta = 90^\circ$

(D) $\vec{A}+\vec{B} = 2\hat{i}+\hat{j}+3\hat{k}$ $|\vec{A}+\vec{B}| = \sqrt{2^2+1+3^2} = \sqrt{14}$

भाग - II

1. $x = t^3 - 3t^2 + 12t + 20$

$v = \frac{dx}{dt} = 3t^2 - 6t + 12$

$t = 0 \Rightarrow v = 12$ m/s

2. $a = \frac{dv}{dt} = 6t - 6$

$t = 0 \Rightarrow a = -6\ m/s^2$

3. $a = 0 \Rightarrow t = 1$ sec.

$v = 3t^2 - 6t + 12 = 9$ m/s

4. $F_{net} = \vec{F}_1 + \vec{F}_2 = 2\hat{i} + 5\hat{j} + 4\hat{k}$

5. $\cos\theta = \frac{\vec{F}_1 \cdot \vec{F}_2}{|\vec{F}_1||\vec{F}_2|} = \left(\frac{3}{5\sqrt{2}}\right)$

6. $F_1 \cos\theta = \frac{\vec{F}_1 \cdot \vec{F}_2}{|\vec{F}_2|} = \frac{6}{5}$

भाग - III

1. $|a| - |b| \le |\vec{a} + \vec{b}| \le |a| + |b|$

2. $R = \sqrt{A_x^2 + B_y^2} = \sqrt{g^2 + 6^2} = 10$

3. सिदान्त पर आधारित

4. सिदान्त पर आधारित

5. दो सदिशो के मध्य कोण निम्न प्रकार दिया जाता है

$\cos\theta = \frac{\vec{A}.\vec{B}}{AB}$

अतः $\cos\theta = \frac{(\hat{i} + \hat{j}).\hat{k}}{|(\hat{i} + \hat{j})||\hat{k}|} = \frac{0}{\sqrt{2} \times 1} = 0$

$\cos\theta = 0 \qquad \theta = \frac{\pi}{2}$

6. सिदान्त पर आधारित

भाग - IV

1. **(i)** यदि $f(x) = e^{-x}$

तब $f'(x) = -e^{-x}$

अतः $f(x) = -f'(x)$

(ii) यदि if $f(x) = e^x$ तब $f'(x) = e^{-x}$

अत $f(x) = f'(x)$

(iii) सिद्धान्त पर आधारित

(iv) सिद्धान्त पर आधारित

(v) सिदान्त पर आधारित

भाग - V

1. $\vec{A} = A\hat{i} \qquad \vec{B} = A\hat{j} \qquad \vec{C} = A\hat{k}$

$\vec{A} + \vec{B} + \vec{C} = A\hat{i} + A\hat{j} + A\hat{k}$

$|\vec{A} + \vec{B} + \vec{C}| = \sqrt{A^2 + A^2 + A^2} = \sqrt{3}\,A$

2. दिया गया $\vec{A} = 3\hat{i} + 4\hat{j}$ $\vec{B} = 7\hat{j} + 24\hat{j}$

माना $\vec{C} = |\vec{C}|\hat{C}$

दिया गया $|\vec{C}| = |\vec{B}| = \sqrt{7^2 + (24)^2}$ = 25 तथा $\hat{C} = \hat{A} = \frac{3\hat{i} + 4\hat{j}}{5}$

$\vec{C} = \frac{25 \times (3\hat{i} + 4\hat{j})}{5}$ $\vec{C} = 15\hat{i} + 20\hat{j}$

3. If $|\vec{A}\vec{B}|$

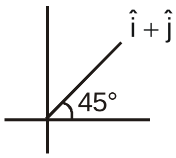
A व B के मध्य कोण शून्य के बराबर होगा अत

So, $\vec{A} \times \vec{B}$ = AB sinθ n = 0

4. समान्तर चतुर्भुज का क्षेत्रफल = $|\vec{A} \times \vec{B}| = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & 0 \\ 2 & 0 & -4 \end{vmatrix} = |-8\hat{i} + 12\hat{j} - 4\hat{k}| = \sqrt{224}$

5. लम्बवत्

6.



7. $\vec{A} = a\hat{i}$ तथा $\vec{B} = a\cos\omega t\hat{i} + a\sin\omega t\hat{j}$

$|\vec{A} + \vec{B}| = \sqrt{3}\,|\vec{A} - \vec{B}|$

$\sqrt{(a + a\cos\omega t)^2 + (a\sin\omega t)^2} = \sqrt{3}\sqrt{(a - a\cos\omega t)^2 + (a\sin\omega t)^2}$

$\Rightarrow 2\cos\frac{\omega t}{2} = \pm\sqrt{3} \times 2\sin\frac{\omega t}{2}$

$\tan\frac{\omega t}{2} = \pm\frac{1}{\sqrt{3}}$

$\frac{\omega t}{2} = n\pi \pm \frac{\pi}{6}$

$\frac{\pi}{12}t = n\pi \pm \frac{\pi}{6}$

t = $(12n \pm 2)$s

= 2s, 10s, 14s ....

HIGH LEVEL PROBLEMS (HLP)
PART - I

1. $g'(x) = \dfrac{(x+0.5)\ \dfrac{d}{dx}\ (x^2-4)-(x^2-4)\dfrac{d}{dx}(x+0.5)}{(x+0.5)^2}$

$= \dfrac{(x+.5)(2x)-(x^2-4)}{(x+0.5)^2}$

$= \dfrac{2x^2+x-x^2+4}{(x+0.5)^2} = \dfrac{x^2+x+4}{(x+0.5)^2}$

2. (a) $\dfrac{d}{dx}$ (uv) $= uv' + vu'$

x = 1 पर $= u(1)\, v'(1) + v(1)\, u'(1)$

$= 2x(-1) + 5(0)$

$= -2$

(b) $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{vu'-uv'}{v^2}$

x = 1 पर $\dfrac{d}{dx}\left(\dfrac{u}{v}\right) = \dfrac{v(1)\ \ u'(1)-u\ \ (1)v'(1)}{[v(1)]^2}$

$= \dfrac{5(0)-(2)(-1)}{(5)^2} = \dfrac{2}{25}$

(c) $\dfrac{d}{dx}\left(\dfrac{v}{u}\right) = \dfrac{uv'-vu'}{u^2} = \dfrac{uv'(1)-v(1)u'(1)}{[u(1)]^2}$

(d) $\dfrac{d}{dx}(7v-2u) = 7v' - 2u'$

x = 1 पर

$= 7\,v'(1) - 2u'\,(1)$

$7(-1) - 2(0) = -7$

3. $S = \dfrac{1+\operatorname{cosec}\ t}{1-\operatorname{cosec}\ t}$

$\dfrac{ds}{dt} = \dfrac{(1-\operatorname{cosec} t)(-\operatorname{cosec} t \cot t)-(1+\operatorname{cosec} t)(+\operatorname{cosec} t \cot t)}{(1-\operatorname{cosec} t)^2}$

$= \dfrac{-2\operatorname{cosec} t \cot t}{(1-\operatorname{cosec} t)^2}$

4. $S = \dfrac{\sin t}{1-\cos t}$

$\dfrac{ds}{dt} = \dfrac{(1-\cos t)\cos t-\sin t(\sin t)}{(1-\cos t)^2}$

$= \dfrac{\cos t\ \ -\ \ (\cos^2 t\ \ +\ \ \sin^2 t)}{(1-\cos\ \ t)^2}$

$= \dfrac{\cos t-1}{(1-\cos t)^2} = \dfrac{1}{\cos t-1}$

5. माना $u = \sin x, y = u^3$: $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = 3u^2 \cos x = 3 \sin^2 x (\cos x)$

6. माना $u = \cos x$

$y = 5 u^{-4}$

$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= 5 (-4) u^{-5} (-\sin x)$

$\frac{dy}{dx} = 20 \sin x \cos^{-5} x$

7. $r = (\csc\theta + \cot\theta)^{-1}$

$r = \frac{1}{\text{cosec}\theta + \cot\theta}$

$\frac{dr}{d\theta} = \frac{(\cot\theta + \text{cosec}\theta)\ \ (0) - 1\ \ (-\text{cosec}^2\theta\ \ -\text{cosec}\theta\ \ \cot\theta)}{(\cot\theta + \text{cosec}\theta)^2}$

$\frac{dr}{d\theta} = \frac{\text{cosec}\theta}{\cot\theta + \text{cosec}\theta}$

8. $r = -(\sec\theta + \tan\theta)^{-1}$

$r = \frac{-1}{\sec\theta + \tan\theta}$

$\frac{dr}{d\theta} = \frac{1\ \ (\sec\theta\ \ \tan\theta + \sec^2\theta)}{(\sec\theta + \tan\theta)^2}$

$= \frac{\sec\theta}{\sec\theta + \tan\theta}$

PART - II

1. (a) $\int \csc x \cot x dx$

$- \csc x + C$

(b) $\int -\csc 5x \cot 5x\, dx$

$= \frac{\csc 5x}{5} + C$

(c) $\int -\pi \csc \frac{\pi}{2} x \cot \frac{\pi x}{2} dx$

$= \frac{+\pi \csc \frac{\pi}{2} x}{\pi/2}$

$= 2 \csc \frac{\pi}{2} x + C$

2. $\int (1 + 2\cos x)^2 dx$

$= \int (1 + 4\cos^2 x + 4\cos x) dx$

$= \int 1.dx + \int 4\cos x\, dx + \int 4\cos^2 x\, dx$

$= x + 4 \sin x + \int 4\left(\frac{\cos 2x + 1}{2}\right) dx$

$= x + 4\sin x + \int 2\cos 2x + \int 2 \cdot dx$

$= x + 4\sin x + \dfrac{2\sin 2x}{2} + 2x$

$= 3x + 4\sin x + \sin 2x + C$

3. $\int \dfrac{x^{1/2}}{2} dx + \int 2x^{-1/2} dx$

$= \dfrac{1}{2}\ \dfrac{x^{1/2+1}}{(1/2+1)} + 2\dfrac{x^{-1/2+1}}{-\frac{1}{2}+1} + c$

$= \dfrac{1}{3}x^{3/2} + 4x^{1/2} + c$

4. $\int 8y\,dy - \int \dfrac{2}{y^{1/4}} dy$

$\dfrac{8y^2}{2} - \dfrac{2\ \ y^{-1/4+1}}{-\frac{1}{4}+1} + c$

$4y^2 - \dfrac{8}{3}y^{3/4} + c$

5. $\int [(2x - (2x)x^{-3}])dx$

$= \int 2x dx - \int 2x^{-2} dx$

$= \dfrac{2x^2}{2} - \dfrac{2x^{-2+1}}{-2+1}$

$= x^2 + \dfrac{2}{x} + c$

10. $\int 4\sec x \tan x \ \ dx - 2\int \sec^2 x \ \ dx$

$= 4\sec x - 2\tan x + c$

11. $\int \dfrac{1}{2} \ \ \csc^2 x \ \ dx - \dfrac{1}{2}\int \csc x \cot x \ \ dx$

$= -\dfrac{1}{2}\cot x + \dfrac{1}{2}\csc x + c$

12. $\int \sin 2x\,dx - \int \csc^2 x\,dx$

$= -\dfrac{\cos\ 2x}{2} + \cot x + c$

13. $\int 2\cos 2x\,dx - \int 3\sin 3x\,dx$

$\dfrac{2\sin 2x}{2} - 3\left(\dfrac{-\cos 3x}{3}\right) + c$

$\sin 2x + \cos 3x + c$

14. $\int 4\sin^2 y\,dy$

$\therefore \quad \sin^2 y = \dfrac{1-\cos 2y}{2}$

$\int 4\left(\dfrac{1-\cos 2y}{2}\right)dy$

$= \int 2\,dy - \int 2\cos 2y\,dy$

$= 2y - \dfrac{2\sin 2y}{2}$

$= 2y - \sin 2y + c$

15. $\int \dfrac{\csc\theta}{\csc\theta - \sin\theta}\,d\theta$

$= \int \dfrac{\frac{1}{\sin\theta}}{\frac{1}{\sin\theta} - \sin\theta}\,d\theta$

$= \int \dfrac{1}{1-\sin^2\theta}\,d\theta$

$= \int \sec^2\theta \cdot d\theta$

$= \tan\theta + c$

16. $\int \dfrac{1}{\sqrt{5s+4}}\,ds$

माना $u = 5s + 4$

$du = 5\,ds$

$ds = \dfrac{du}{5}$

$= \dfrac{1}{5}\int u^{-1/2}du$

$= \dfrac{1}{5}\,\dfrac{u^{-1/2}+1}{-1/2+1} + c$

$= \dfrac{2}{5}\sqrt{u} + c = \dfrac{2}{5}\sqrt{5s+4} + c$

17. $\int 3y\sqrt{7-3y^2}\,dy$

माना $7 - 3y^2 = t^2$

दोनो तरफ y के सापेक्ष अवकलन करने पर

$-6y\,dy = 2t\,dt$

$= 3y\,dy = -t\,dt$

$= \int -t^2\,dt$

$= -\dfrac{t^3}{3} + c$

$= -\dfrac{(7-3y^2)^{3/2}}{3} + c$

18. $\int \sin^5 \frac{x}{3} \cos \frac{x}{3} dx$

माना $\sin \frac{x}{3} = t$

$\cos \frac{x}{3} \frac{dx}{3} = dt$

$\cos \frac{x}{3} \; dx = 3 \; dt$

$= \int t^5 3dt = 3 \frac{t^6}{6} + c$

$= \frac{\sin^6 \frac{x}{3}}{2} + c$

19. $\int \tan^7 \left(\frac{x}{2}\right) \sec^2 \frac{x}{2} \cdot dx$

माना $\tan \frac{x}{2} = t$

$\sec^2 \frac{x}{2} \cdot \frac{dx}{2} = dt$

$\sec^2 \frac{x}{2} \; dx = 2 \; dt$

$\int t^7 \; 2dt = 2 \frac{t^8}{8} + c$

$= \frac{\tan^8 x/2}{4} + c$

20. $\int r^2 \left(\frac{r^3}{18} - 1\right)^5 dr$

माना $\frac{r^3}{18} - 1 = t$

$\frac{3r^2}{18} dr = dt$

$r^2 dr = 6dt$

$= \int t^5 6 \, dt$

$= 6 \; \frac{t^6}{6} + c$

$= \left(\frac{r^3}{18} - 1\right)^6 + c$

21. $\int r^4 \left(7 - \frac{r^5}{10}\right)^3 dr$

माना $7 - \frac{r^5}{10} = t$

$- \frac{5r^4}{10} \; dr = dt$

$r^4 \; dr = -2 \; dt$

$\int t^3(-2dt)$

$= -2\,\frac{t^4}{4} + c$

$= -\frac{\left(7-\frac{r^5}{10}\right)^4}{2} + c$

22. $\int x^{\frac{1}{3}} \sin(x^{\frac{4}{3}} - 8)\, dx$

माना $x^{4/3} - 8 = t$

$\frac{4}{3}\, x^{1/3}\, dx = dt$

$x^{1/3}\, dx = \frac{4}{3} dt$

$\int \frac{3}{4} \sin t dt$

$= -\frac{3}{4} \cos t + c$

$= -\frac{3}{4} \cos (x^{4/3} - 8) + c$

23. $\int \csc\left(\frac{\upsilon-\pi}{2}\right)\cot\left(\frac{\upsilon-\pi}{2}\right) d\upsilon$

माना $\frac{\upsilon-\pi}{2} = t \;\Rightarrow\; d\upsilon = 2dt$

$\int \csc t \cot t\; 2dt = -2 \csc t$

$= -2 \csc\left(\frac{\upsilon-\pi}{2}\right) + c$

24. $\int \sqrt{\cot y}\;\; \csc^2 y\;\, dy$

माना $\cot y = t$

$-\csc^2 y\, dy = dt$

$-\int \sqrt{t}\, dt$

$-\frac{2t^{3/2}}{3} + c$

$= -\frac{2}{3}\,(\cot y)^{3/2} + c$

25. $\int \frac{\sec z \tan z}{\sqrt{\sec z}}\, dr$

माना $\sec z = t$

$\sec z \tan z\, dz = dt$

$\int \frac{1}{\sqrt{t}}\, dt$

$= 2\sqrt{t} + c$

$= 2\sqrt{\sec z} + c$

26. $\int \frac{1}{t^2}\cos\left(\frac{1}{t}-1\right)dt$

माना $\frac{1}{t}-1 = x$

$= -\frac{1}{t^2}\,dt = dx$

$\int \cos x(-dx)$

$= -\sin x + c$

$= -\sin\left(\frac{1}{t}-1\right) + c$

27. $\int_{1/2}^{3/2} (-2x+4)\ dx$

$= \left[-x^2+4x\right]_{1/2}^{3/2}$

$= \left[-\left(\frac{3}{2}\right)^2+4\left(\frac{3}{2}\right)\right] - \left[-\frac{1}{4}+4\times\frac{1}{2}\right]$

= 2 वर्ग इकाई

28. $\int_{-2}^{1} |x|dx$

$\int_{-2}^{0} -x\,dx + \int_{0}^{1} dx = \left[-\frac{x^2}{2}\right]_{-2}^{0} + \left[\left(\frac{x^2}{2}\right)\right]_{0}^{1}$

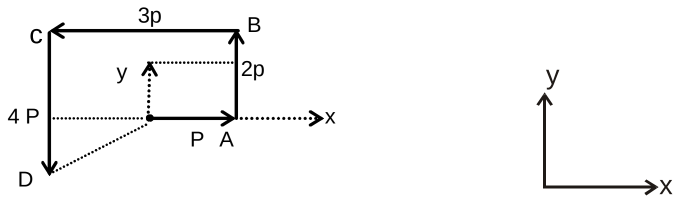
= 2 + 1/2 = 2.5 वर्ग इकाई

29. $\int_{0}^{\pi/2} \theta^2\,d\theta \Rightarrow \left[\frac{\theta^3}{3}\right]_{0}^{\pi/2} \Rightarrow \frac{\pi^3}{24}$

30. $\int_{0}^{3b} x^2dx \Rightarrow \left[\frac{x^3}{3}\right]_{0}^{3b} = 9b^3$

PART - III

1.



माना O मूल बिन्दु है तथा x अक्ष तथा y अक्ष के अनुदिश एकांक सदिश क्रमशः $\hat{i}$ तथा $\hat{j}$ हैं

अतः $\overrightarrow{OA} = P\hat{i}$

$\overrightarrow{AB} = 2P\hat{j}$

$\overrightarrow{BC} = -3P\hat{i}$

$\overrightarrow{CD} = -\ 4P\hat{j}$

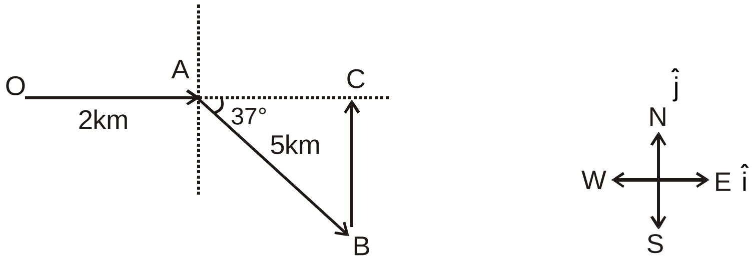
परिणामी बल $\vec{R} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$

$= P\hat{i} + 2p\hat{i} - 3p\hat{i} - 4p\hat{j}$

$\vec{R} = -2p\hat{i} - 2p\hat{j}$

$|\vec{R}| = 2\sqrt{2}$ P **Ans.**

2.



नाव का अन्तिम विस्थापन = $6\hat{i}$, चित्र से

$\overrightarrow{OA} = 2\hat{i}$

$\overrightarrow{BC} = 5\cos37°\,\hat{i} - 5\sin37°\,\hat{j} = 4\hat{i} - 3\hat{j}$

चित्र से

$\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AB} + \overrightarrow{BC}$

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OA} - \overrightarrow{AB}$

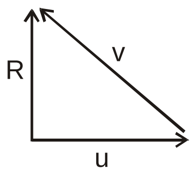
$= 6\hat{i} - 2\hat{i} - 4\hat{i} + 3\hat{j}$

$\overrightarrow{BC} = 3\hat{j}$

= 3 km उत्तर में

3. दिया है $\vec{R} = \vec{u} + \vec{v}$ तथा $\vec{R} \perp \vec{u}$

तथा $|\vec{R}| = \frac{|\vec{v}|}{2}$



माना $|\vec{v}| = x$ then तो $|\vec{R}| = x/2$

चित्र से $u^2 = v^2 - R^2 = x^2 - \frac{x^2}{4} \Rightarrow u = \frac{\sqrt{3}}{2}x$

हम जानते हैं कि

$R^2 = u^2 + v^2 + 2uv\cos\theta$

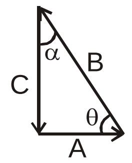
$\frac{x^2}{4} = \frac{3}{4}x^2 + x^2 + 2\left(\frac{\sqrt{3}}{2}x\right)(x)\cos\theta$

$\Rightarrow \quad \cos\theta = -\frac{\sqrt{3}}{2}$

$\theta = 150°$

4. दिया है $|\vec{A}| = 5\vec{N}$, $|\vec{B}| = 13N$ & $|\vec{C}| = 12N$

$\vec{A}$ & $\vec{B}$ के बीच कोण से लिए, चित्र से $\sin\alpha = \frac{5}{13}$



$\alpha = 23°$

अतः कोण $\theta = 90 - 23$

अतः $\vec{A} \& \vec{B}$ के बीच कोण $= 180 - \theta = 180 - 90 + 23 = 90 + 23 = 113°$

5. दी गई शर्त के अनुसार

$|\vec{A} + \vec{B}| = |\vec{A}| = |\vec{B}| = x$

$A^2 + B^2 + 2AB\cos\theta = A^2$

$x^2 + x^2 + 2x^2\cos\theta = x^2$

$\cos\theta = -1/2$

$\theta = 120°$

$|\vec{A} - \vec{B}| = |\vec{A}|$

$A^2 + B^2 - 2AB\cos\theta = A^2$

$x^2 + x^2 - 2x^2\cos\theta' = x^2$

$\cos\theta' = 1/2$

$\theta' = 60°$

6. माना $\vec{P}$ तथा $\vec{Q}$ के बीच कोण θ है तो

$R^2 = |\vec{P}+\vec{Q}|^2 = P^2 + Q^2 + 2PQ\cos\theta$ ....(i)

यदि $\vec{Q}$ को दुगना कर दें तो, $\vec{R}$ दुगना हो जायेगा अर्थात् $2\vec{Q}$ तथा $\vec{P}$ के परिणाम का परिमाण 2R हो जायेगा।

$(2R)^2 = P^2 + (2Q)^2 + 2P(2Q)\cos\theta$

इससे, $4R^2 = P^2 + 4Q + 4PQ\cos\theta$ ....(ii)

जब $\vec{Q}$ को विपरीत कर देते है तो $\vec{R}$ दुगना हो जाता है। अतः $\vec{P}$ तथा $(-\vec{Q})$ के परिणाम का परिमाण 2R है।

तो, $(2R)^2 = P^2 + Q^2 + 2PQ\cos(180^o - \theta)$.

इससे $4R^2 = P^2 + Q^2 - 2PQ\cos\theta$ ...(iii)

समी० (i) – समी० (iii) से $PQ\cos\theta = \frac{-3R^2}{4}$ ...(iv)

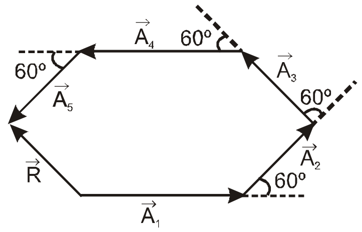
समी० (i) + समी० (iii) से $P^2 + Q^2 = \frac{5R^2}{2}$ ....(v)

समी० (ii) + समी० (iv) से $P^2 + 4Q^2 = 7R^2$ ...(vi)

समी० (v) तथा समी० (vi) को हल करने पर $Q = \sqrt{\frac{3}{2}}R$ तथा $P = R$

अतः $P : Q : R = \sqrt{2} : \sqrt{3} : \sqrt{2}$ **Ans.**

7. सदिश आरेख में प्रत्येक सदिश अपने पास वाले सदिश से 60º का कोण बना रहा है। प्रत्येक सदिश की पूंछ को मूल बिन्दु पर रखते है। सदिशों को इस तरह से रखने पर $\vec{A}_1$ तथा $\vec{A}_4$ समान तथा विपरीत है। इसी प्रकार $\vec{A}_2$ तथा $\vec{A}_5$ समान तथा विपरीत प्राप्त होते है।



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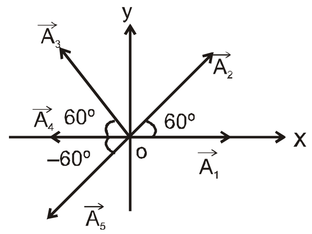


$\vec{A}_1 + \vec{A}_4 = \vec{0}$ तथा $\vec{A}_2 + \vec{A}_5 = \vec{0}$

अतः $\vec{R} = \vec{A}_1 + \vec{A}_2 + \vec{A}_3 + \vec{A}_4 + \vec{A}_5$

$= \vec{A}_3 = (|\vec{A}_3| \cos 60^o)(-\hat{i}) + (|\vec{A}_3| \sin 60^o)\,\hat{j}$

$|\vec{A}_3| = 1$, रखने पर, $\vec{A}_3 = \frac{1}{2}(-\hat{i} + \sqrt{3}\hat{j})$

8. चींटी द्वारा लगाये गये बल

$\vec{F}_1 = 3\,(\cos 30^o\,\hat{i} + \sin 30^o\,\hat{j})$

$\vec{F}_2 = -1\,\hat{i}$

$\vec{F}_3 = 2\cos 45^o\,\hat{i} + 2\sin 45^o\,(-\hat{j})$

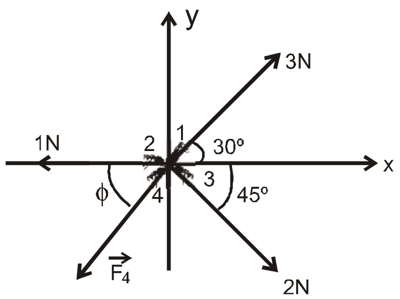
तथा $(= F) = x\,\hat{i} + y\,\hat{j}$, say.

अतः गैंहू साम्यावस्था में है अतः इस पर परिणामी बल

$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 = \vec{0}$

उपरोक्त बलों के मान रखने पर

$$\hat{i}\left(\frac{3\sqrt{3}}{2} - 1 + \sqrt{2} + x\right) + \left(\frac{3}{2} - \sqrt{2} + y\right)\hat{j} = 0$$



$\hat{i}$ तथा $\hat{j}$ के गुणांको की तुलना करने पर,

$x = -\left(\frac{3\sqrt{3}}{2} + \sqrt{2} - 1\right)$ and तथा $y = -\left(\frac{3}{2} - \sqrt{2}\right)$

अतः 4th चींटी द्वारा लगाये गये बल का परिमाण है

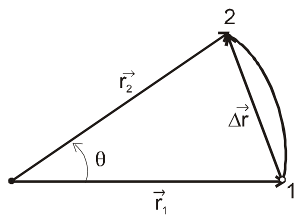
$$F_4 = \sqrt{x^2 + y^2} = \sqrt{\left(\frac{3\sqrt{3}}{2} + \sqrt{2} - 1\right)^2 + \left(\frac{3}{2} - \sqrt{2}\right)^2}$$

तथा इसका –ve x-अक्ष से बनाया गया कोण $\phi = \tan^{-1}\left[\frac{3 - 2\sqrt{2}}{3\sqrt{3} + 2\sqrt{2} - 2}\right]$ है। यह चित्र में प्रदर्शित है।

9. जब चींटी स्थिति 1 से स्थिति 2, की तरफ चलती है। विस्थापन $\vec{s} = \Delta\vec{r}$ स्थिति सदिश में परिवर्तन

अतः $\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$, तो विस्थापन का परिमाण

$|\Delta\vec{r}| = \sqrt{r_1^2 + r_2^2 - 2r_1r_2\cos\theta}$

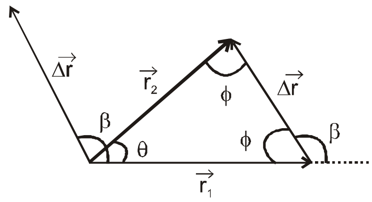


चूंकि चींटी वृत्ताकार पथ में घूमती है।

$r_1 = r_2 = R$

अतः $|\Delta\vec{r}| = 2R\sin\frac{\theta}{2}$, $\Delta\vec{r}$ की दिशा

$\beta = \frac{\pi}{2} + \frac{\theta}{2}$ है। यह चित्र में प्रदर्शित है।



10. माना एक सीधी रेखा PQ प्रदर्शित की जाती है।

$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

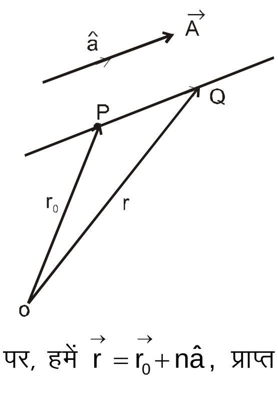
जहाँ $\overrightarrow{OQ} = \vec{r}$ तथा $\overrightarrow{OP} = \vec{r}_0$ (दिया है)

$\overrightarrow{PQ} = \vec{r} - \vec{r}_0$

चूंकि $\overrightarrow{PQ} \parallel \vec{A}$ (दिया है)

$\overrightarrow{PQ} = PQ.\ \hat{a}$, जहाँ $\hat{a} = \frac{\vec{A}}{A}$

अतः $\vec{r} - \vec{r}_0 = \frac{PQ}{A}\ \hat{a}$



$\vec{r} = \vec{r}_0 + \frac{PQ}{A}\ \hat{a}$ तथा $\frac{PQ}{A} = n$ रखने पर, हमें $\vec{r} = \vec{r}_0 + n\hat{a}$, प्राप्त होता है।

11. $\vec{A}+\vec{B}=\vec{R}$, say,

चूंकि $|\vec{R}|^2 = \vec{R}.\vec{R} = (\vec{A}+\vec{B}).(\vec{A}+\vec{B})$

गुणनफल करने पर

$$|\vec{R}|^2 = \vec{A}.\vec{A}+\vec{A}.\vec{B}+\vec{B}.\vec{A}+\vec{\mathbf{B}}.\vec{B}$$

$\vec{A}.\vec{A} = A^2$, $\vec{A}.\vec{B}=\vec{B}.\vec{A}$ तथा $\vec{B}.\vec{B} = |\vec{B}|^2$ रखने पर

$$|\vec{R}|^2 = |\vec{A}|^2 + |\vec{B}|^2 + 2\vec{A}.\vec{B}$$

अतः $|\vec{R}| = |\vec{A}+\vec{B}| = \sqrt{|\vec{A}|^2 + |\vec{B}|^2 + 2|\vec{A}||\vec{B}|\cos\theta}$

12. $\vec{A} = t\hat{i} - \sin\pi t\hat{j} + t^2\hat{k}$ का दोंनो तरफ अवकलन करने पर

we have $\frac{d\vec{A}}{dt} = \frac{d}{dt}(t\hat{i} - \sin\pi t\hat{j} + t^2\hat{k})$

$$= \frac{d}{dt}(t)\hat{i} - \frac{d}{dt}(\sin\pi t)\hat{j} + \frac{d}{dt}(t^2)\hat{k}$$

$$= \hat{i} - \pi\cos\pi t\hat{j} + 2t\hat{k}$$

t =1 रखने पर $\left.\frac{d\vec{A}}{dt}\right|_{t=1} = \hat{i} - (\pi\cos\pi)\hat{j} + 2(1)\hat{k}$

$$= \hat{i} + \pi\hat{j} + 2\hat{k}$$

13. हम देखते है

स्थिति : $s = 5\cos t$

वेग : $v = \frac{ds}{dt} = \frac{d}{dt}(5\cos t) = 5\frac{d}{dt}(\cos t) = -5\sin t$

त्वरण : $a = \frac{dv}{dt} = \frac{d}{dt}(-5\sin t) = -5\frac{d}{dt}(\sin t) = -5\cos t$

14. (a) पानी का आयतन नियत रहता है।

$v = \pi(4R^2)h - \pi R^2 x$

$0 = 16\pi R^2 \frac{dh}{dt} - \pi R^2 \frac{dx}{dt} = 0$

$$\frac{dh}{dt} = \frac{1}{16}v$$

(b) गिली सतह का क्षेत्रफल ℓ

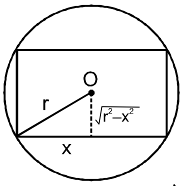
$A = \pi R^2 + 2\pi R\frac{d\ell}{dt}$

पानी के अन्दर की लम्बाई ℓ

$\frac{dA}{dt} = 0 + 2\pi R\frac{d\ell}{dt}$ $\quad\because\quad \frac{d\ell}{dt} = \frac{dx}{dt} + \frac{dh}{dt}$

$\frac{dA}{dt} = 2\pi R\left[v + \frac{v}{16}\right] = \frac{17\pi\ Rv}{8}$ **Ans.** **(a)** $\frac{dh}{dt} = \frac{v}{15}$ **(b)** $\frac{32\pi Rv}{15}$

15.



माना आयत का क्षेत्रफल = A

A = 8 × $\frac{1}{2}$ × x × $\sqrt{r^2 - x^2}$ = 4 x $\sqrt{r^2 - x^2}$

उच्चिष्ठ के लिए

$$\frac{dA}{dx} = 0\,, \quad \frac{dA}{dx} = 4\left(\frac{x(-2x)}{2\sqrt{r^2 - x^2}} + \sqrt{r^2 - x^2}\right)$$

$$0 = \frac{4(r^2 - 2x^2)}{\sqrt{r^2 - x^2}}$$

or x = $\frac{r}{\sqrt{2}}$

For x = $\frac{r}{\sqrt{2}}$, $\frac{d^2A}{dx^2} < 0$

∴ x = $\frac{r}{\sqrt{2}}$ के लिए क्षेत्रफल अधिकतम होगा

तथा $A_{max} = \frac{4r}{\sqrt{2}}\sqrt{r^2 - \left(\frac{r}{\sqrt{2}}\right)^2}$

$A_{max} = 2r^2$.