RECTILINEAR MOTION

Contents

JEE (ADVANCED) SYLLABUS

Kinematics in one Dimensions.

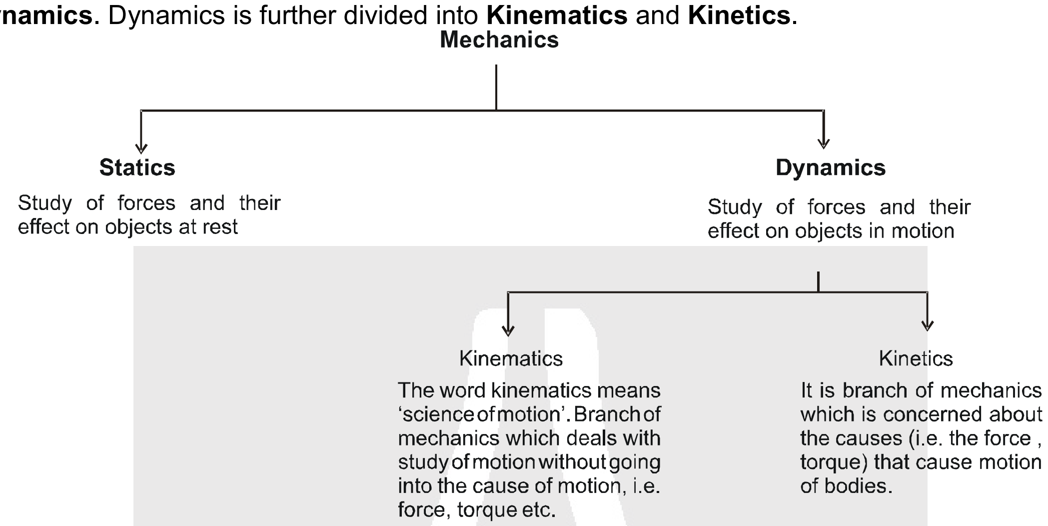
JEE (MAIN) SYLLABUS

Motion in a straight line : Position time graph, speed and velocity. Uniform and non-uniform motion, average speed and instantaneous velocity Uniformly acceleration motion, velocity-time, position-time graphs, relations for uniformly accelerated motion.

RECTILINEAR MOTION

1. MECHANICS

Mechanics is the branch of physics which deals with the cause and effects of motion of a particle, rigid objects and deformable bodies etc. Mechanics is classified under two streams namely **Statics** and **Dynamics**. Dynamics is further divided into **Kinematics** and **Kinetics**.



2. MOTION AND REST

Motion is a combined property of the object and the observer. There is no meaning of rest or motion without the observer. Nothing is in absolute rest or in absolute motion.

An object is said to be in motion with respect to a observer, if its position changes with respect to that observer. It may happen by both ways either observer moves or object moves.

3. RECTILINEAR MOTION

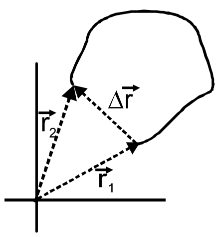
Rectilinear motion is motion, along a straight line or in one dimension. It deals with the kinematics of a particle in one dimension.

3.1 Position

The position of a particle refers to its location in the space at a certain moment of time. It is concerned with the question – "where is the particle at a particular moment of time?"

3.2 Displacement

The change in the position of a moving object is known as displacement. It is the vector joining the initial position ($\vec{r}_1$) of the particle to its final position ($\vec{r}_2$) during an interval of time.



$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1$$

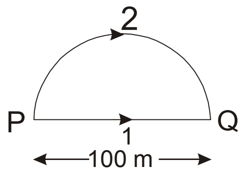
Displacement can be negative positive or zero.

3.3 Distance

The length of the actual path travelled by a particle during a given time interval is called as distance. The distance travelled is a scalar quantity which is quite different from displacement. In general, the distance travelled between two points may not be equal to the magnitude of the displacement between the same points.

Solved Example

Example 1. Ram takes path 1 (straight line) to go from P to Q and Shyam takes path 2 (semicircle).



(a) Find the distance travelled by Ram and Shyam?

(b) Find the displacement of Ram and Shyam?

Solution :

(a) Distance travelled by Ram = 100 m
Distance travelled by Shyam = π(50 m) = 50π m

(b) Displacement of Ram = 100 m
Displacement of Shyam = 100 m

3.4 Average Velocity (in an interval) :

The average velocity of a moving particle over a certain time interval is defined as the displacement divided by the lapsed time.

Average Velocity = $\dfrac{\text{displacement}}{\text{time interval}}$

for straight line motion, along x-axis, we have

$$v_{av} = \bar{v} = \langle v \rangle = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i}$$

The dimension of velocity is $[LT^{-1}]$ and its SI unit is m/s.
The average velocity is a vector in the direction of displacement. For motion in a straight line, directional aspect of a vector can be taken care of by +ve and -ve sign of the quantity.

3.5 Instantaneous Velocity (at an instant) :

The velocity at a particular instant of time is known as instantaneous velocity. The term "velocity" usually means instantaneous velocity.

$$V_{inst.} = \lim_{\Delta t \to 0}\left(\frac{\Delta x}{\Delta t}\right) = \frac{dx}{dt}$$

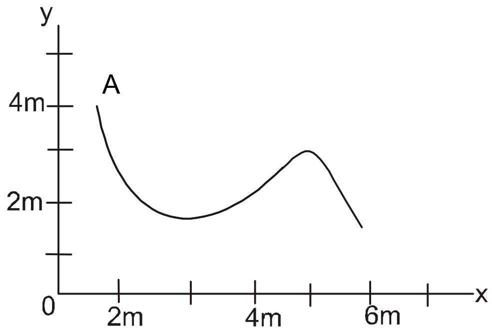
In other words, the instantaneous velocity at a given moment (say, t) is the limiting value of the average velocity as we let Δt approach zero. The limit as $\Delta t \to 0$ is written in calculus notation as dx/dt and is called the derivative of x with respect to t.

Note :

- The magnitude of instantaneous velocity and instantaneous speed are equal.
- The determination of instantaneous velocity by using the definition usually involves calculation of derivative. We can find $v = \dfrac{dx}{dt}$ by using the standard results from differential calculus.
- Instantaneous velocity is always tangential to the path.

Solved Example

Example 1. A particle starts from a point A and travels along the solid curve shown in figure. Find approximately the position B of the particle such that the average velocity between the positions A and B has the same direction as the instantaneous velocity at B.

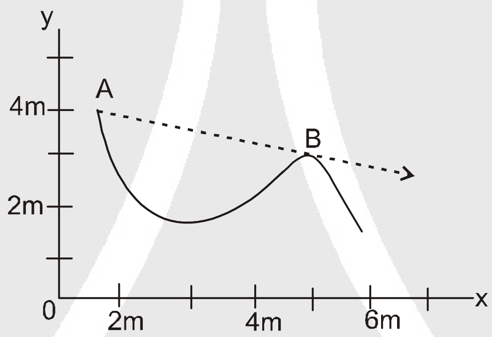


Answer : x = 5m, y = 3m

Solution : The given curve shows the path of the particle starting at y = 4 m.

Average velocity = $\dfrac{\text{displacement}}{\text{time taken}}$ **;** where displacement is straight line distance between points.

Instantaneous velocity at any point is the tangent drawn to the curve at that point.



Now, as shown in the graph, line AB shows displacement as well as the tangent to the given curve. Hence, point B is the point at which direction of AB shows average as well as instantaneous velocity.

3.6 Average Speed (in an interval)

Average speed is defined as the total path length travelled divided by the total time interval during which the motion has taken place. It helps in describing the motion along the actual path.

$$\textbf{Average Speed} = \frac{\text{distance travelled}}{\text{time interval}}$$

The dimension of velocity is [LT^{-1}] and its SI unit is m/s.

Note :

- Average speed is always positive in contrast to average velocity which being a vector, can be positive or negative.
- If the motion of a particle is along a straight line and in same direction then, average velocity = average speed.
- Average speed is, in general, greater than the magnitude of average velocity.

Solved Example

Example 1. In the example 1, if Ram takes 4 seconds and Shyam takes 5 seconds to go from P to Q, find
(a) Average speed of Ram and Shyam?
(b) Average velocity of Ram and Shyam?

Solution :

(a) Average speed of Ram = $\frac{100}{4}$ m/s = 25 m/s

Average speed of Shyam = $\frac{50\pi}{5}$ m/s = 10π m/s

(b) Average velocity of Ram = $\frac{100}{4}$ m/s = 25 m/s (From P to Q)

Average velocity of Shyam = $\frac{100}{5}$ m/s = 20 m/s (From P to Q)

Example 2. A particle travels half of total distance with speed v_1 and next half with speed v_2 along a straight line. Find out the average speed of the particle?

Solution : Let total distance travelled by the particle be 2s.

Time taken to travel first half = $\frac{s}{v_1}$

Time taken to travel next half = $\frac{s}{v_2}$

Average speed = $\frac{\text{Total distance covered}}{\text{Total time taken}} = \frac{2s}{\frac{s}{v_1}+\frac{s}{v_2}} = \frac{2v_1v_2}{v_1+v_2}$ (harmonic progression)

Example 3. A person travelling on a straight line moves with a uniform velocity v_1 for some time and with uniform velocity v_2 for the next equal time. The average velocity v is given by

Answer : $v = \frac{v_1+v_2}{2}$ (Arithmetic progression)

Solution :

As shown, the person travels from A to B through a distance S, where first part S_1 is travelled in time t/2 and next S_2 also in time t/2.

So, according to the condition : $v_1 = \frac{S_1}{t/2}$ and $v_2 = \frac{S_2}{t/2}$

Average velocity = $\frac{\text{Total displacement}}{\text{Total time taken}} = \frac{S_1+S_2}{t} = \frac{\frac{v_1t}{2}+\frac{v_2t}{2}}{t} = \frac{v_1+v_2}{2}$

3.7 Average acceleration (in an interval):

The average acceleration for a finite time interval is defined as :

Average acceleration = $\frac{\text{change in velocity}}{\text{time interval}}$

Average acceleration is a vector quantity whose direction is same as that of the change in velocity.

$$\vec{a}_{av} = \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_f - \vec{v}_i}{\Delta t}$$

Since for a straight line motion the velocities are along a line, therefore

$$a_{av} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_i}{t_f - t_i}$$

(where one has to substitute v_f and v_i with proper signs in one dimensional motion)

3.8 Instantaneous Acceleration (at an instant):

The instantaneous acceleration of a particle is its acceleration at a particular instant of time. It is defined as the derivative (rate of change) of velocity with respect to time. We usually mean instantaneous acceleration when we say "acceleration". For straight motion we define instantaneous acceleration as :

$$a = \frac{dv}{dt} = \lim_{\Delta t \to 0}\left(\frac{\Delta v}{\Delta t}\right) \quad \text{and in general} \quad \vec{a} = \frac{d\vec{v}}{dt} = \lim_{\Delta t \to 0}\left(\frac{\Delta \vec{v}}{\Delta t}\right)$$

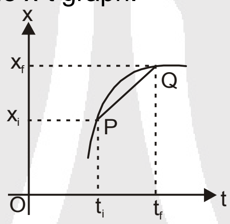
The dimension of acceleration is $[LT^{-2}]$ and its SI unit is m/s^2.

4. GRAPHICAL INTERPRETATION OF SOME QUANTITIES

4.1 Average Velocity

If a particle passes a point P (x_i) at time $t = t_i$ and reaches Q (x_f) at a later time instant $t = t_f$, its average velocity in the interval PQ is $V_{av} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i}$

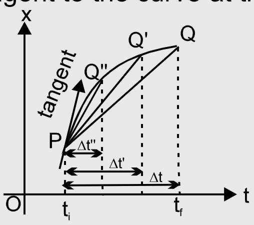
This expression suggests that the average velocity is equal to the slope of the line (chord) joining the points corresponding to P and Q on the x-t graph.



4.2 Instantaneous Velocity

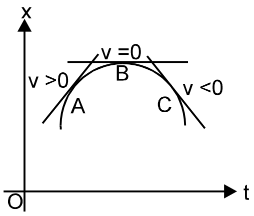
Consider the motion of the particle between the two points P and Q on the x-t graph shown. As the point Q is brought closer and closer to the point P, the time interval between PQ (Δt, $\Delta t'$, $\Delta t''$,......) get progressively smaller. The average velocity for each time interval is the slope of the appropriate dotted line (PQ, PQ′, PQ″......).

As the point Q approaches P, the time interval approaches zero, but at the same time the slope of the dotted line approaches that of the tangent to the curve at the point P. As $\Delta t \to 0$, V_{av} $(=\Delta x/\Delta t) \to V_{inst.}$



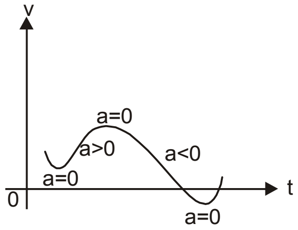
Geometrically, as $\Delta t \to 0$, chord PQ $\to$ tangent at P.

Hence the instantaneous velocity at P is the slope of the tangent at P in the x – t graph. When the slope of the x – t graph is positive, v is positive (as at the point A in figure). At C, v is negative because the tangent has negative slope. The instantaneous velocity at point B (turning point) is zero as the slope is zero.



4.3 Instantaneous Acceleration :

The derivative of velocity with respect to time is the slope of the tangent in velocity time (v–t) graph.



Solved Example

Example 1. Position of a particle as a function of time is given as $x = 5t^2 + 4t + 3$. Find the velocity and acceleration of the particle at t = 2 s?

Solution : Velocity; $v = \frac{dx}{dt} = 10t + 4$

At $t = 2$ s

$v = 10(2) + 4$

$v = 24$ m/s

Acceleration; $a = \frac{d^2x}{dt^2} = 10$

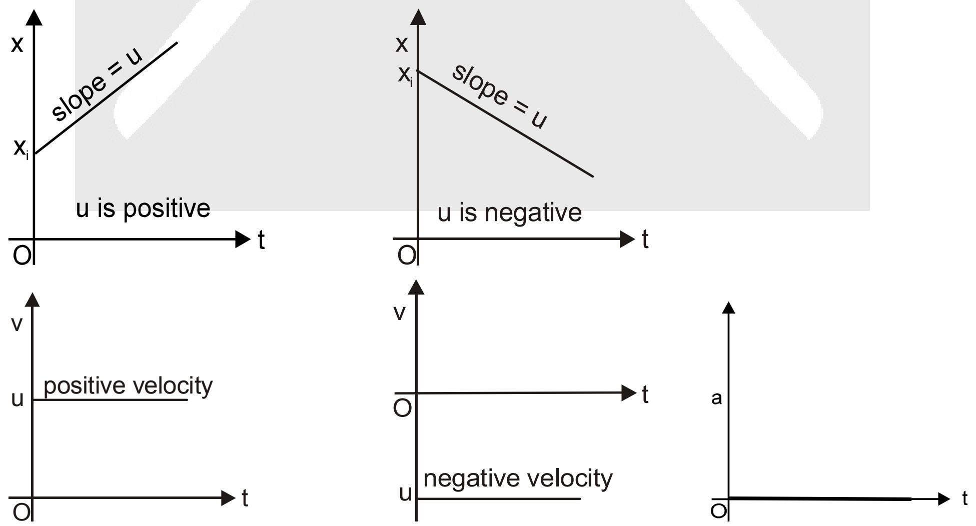
Acceleration is constant, so at t = 2 s $a = 10$ m/s^2

5. MOTION WITH UNIFORM VELOCITY

Consider a particle moving along x-axis with uniform velocity u starting from the point $x = x_i$ at t = 0.

Equations of x, v, a are : $x(t) = x_i + ut$; $v(t) = u$; $a(t) = 0$

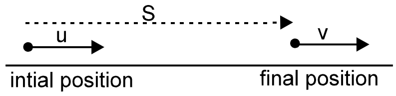
- x-t graph is a straight line of slope u through x_i.
- as velocity is constant, v – t graph is a horizontal line.
- a-t graph coincides with time axis because a = 0 at all time instants.



6. UNIFORMLY ACCELERATED MOTION :

If a particle is accelerated with constant acceleration in an interval of time, then the motion is termed as uniformly accelerated motion in that interval of time.

For uniformly accelerated motion along a straight line (x-axis) during a time interval of t seconds, the following important results can be used.



(i) $a = \frac{v - u}{t}$

(ii) $V_{av} = \frac{v + u}{2}$

(iii) $S = (v_{av})t$

(iv) $S = \left(\frac{v + u}{2}\right)t$

(v) $v = u + at$

(vi) $s = ut + 1/2\ at^2$; $s = vt - 1/2\ at^2$
$x_f = x_i + ut + 1/2\ at^2$

(vii) $v^2 = u^2 + 2as$

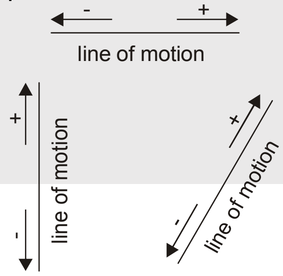
(viii) $s_n = u + a/2\ (2n - 1)$

u = initial velocity (at the beginning of interval)
a = acceleration
v = final velocity (at the end of interval)
s = displacement $(x_f - x_i)$
x_f = final coordinate (position)
x_i = initial coordinate (position)
s_n = displacement during the n^{th} sec

7. DIRECTIONS OF VECTORS IN STRAIGHT LINE MOTION

In straight line motion, all the vectors (position, displacement, velocity & acceleration) will have only one component (along the line of motion) and there will be only two possible directions for each vector.

- For example, if a particle is moving in a horizontal line (x-axis), the two directions are right and left. Any vector directed towards right can be represented by a positive number and towards left can be represented by a negative number.
- For vertical or inclined motion, upward direction can be taken +ve and downward as –ve



- For objects moving vertically near the surface of the earth, the only force acting on the particle is its weight (mg) i.e. the gravitational pull of the earth. Hence acceleration for this type of motion will always be $a = -g$ i.e., $a = -9.8\ m/s^2$ (–ve sign, because the force and acceleration are directed downwards, If we select upward direction as positive).

Note :
- If acceleration is in same direction as velocity, then speed of the particle increases.
- If acceleration is in opposite direction to the velocity then speed decreases i.e., the particle slows down. This situation is known as retardation.

Solved Example

Example 1. A particle moving rectilinearly with constant acceleration is having initial velocity of 10 m/s. After some time, its velocity becomes 30 m/s. Find out velocity of the particle at the mid point of its path?

Solution : Let the total distance be 2x.

$\therefore$ distance upto midpoint = x

Let the velocity at the mid point be v and acceleration be a.

From equations of motion

$v^2 = 10^2 + 2ax$ .....(1)

$30^2 = v^2 + 2ax$ .....(2)

(2) – (1) gives

$v^2 - 30^2 = 10^2 - v^2$

$\Rightarrow$ $v^2 = 500$ $\Rightarrow$ $v = 10\sqrt{5}$ m/s

Example 2. Mr. Sharma brakes his car with constant acceleration from a velocity of 25 m/s to 15 m/s over a distance of 200 m.

(a) How much time elapses during this interval?

(b) What is the acceleration?

(c) If he has to continue braking with the same constant acceleration, how much longer would it take for him to stop and how much additional distance would he cover?

Solution :

(a) We select positive direction for our coordinate system to be the direction of the velocity and choose the origin so that $x_i = 0$ when the braking begins. Then the initial velocity is $u_x = +25$ m/s at t = 0, and the final velocity and position are $v_x = +15$ m/s and x = 200 m at time t. Since the acceleration is constant, the average velocity in the interval can be found from the average of the initial and final velocities.

$\therefore$ $v_{av,\,x} = \frac{1}{2}(u_x + v_x) = \frac{1}{2}(15 + 25) = 20$ m/s.

The average velocity can also be expressed as $v_{av,\,x} = \frac{\Delta x}{\Delta t}$. With $\Delta x = 200$ m

and $\Delta t = t - 0$, we can solve for t : $t = \frac{\Delta x}{v_{av,x}} = \frac{200}{20} = 10$ s.

(b) We can now find the acceleration using $v_x = u_x + a_x t$

$a_x = \frac{v_x - u_x}{t} = \frac{15 - 25}{10} = -1$ m/s^2.

The acceleration is negative, which means that the positive velocity is becoming smaller as brakes are applied (as expected).

(c) Now with known acceleration, we can find the total time for the car to go from velocity $u_x = 25$ m/s to $v_x = 0$. Solving for t, we find

$t = \frac{v_x - u_x}{a_x} = \frac{0 - 25}{-1} = 25$ s.

The total distance covered is $x = x_i + u_x t + \frac{1}{2} a_x t^2$

$= 0 + (25)(25) + 1/2\,(-1)(25)^2 = 625 - 312.5 = 312.5$ m.

Additional distance covered = 312.5 – 200 = 112.5 m.

Example 3. A police inspector in a jeep is chasing a pickpocket an a straight road. The jeep is going at its maximum speed v (assumed uniform). The pickpocket rides on the motorcycle of a waiting friend when the jeep is at a distance d away, and the motorcycle starts with a constant acceleration a. Show that the pick pocket will be caught if $v \ge \sqrt{2ad}$.

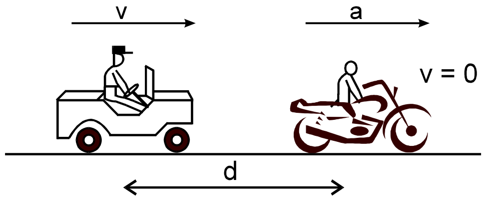
Solution : Suppose the pickpocket is caught at a time t after motorcycle starts. The distance travelled by the motorcycle during this interval is

$$s = \frac{1}{2}at^2 \qquad ......(1)$$

During this interval the jeep travels a distance

$$s + d = vt \qquad ......(2)$$

By (1) and (2),

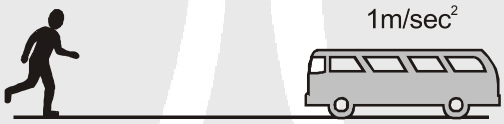


$$\frac{1}{2}at^2 + d = vt \quad \text{or,} \quad t = \frac{v \pm \sqrt{v^2 - 2ad}}{a}$$

The pickpocket will be caught if t is real and positive. This will be possible if

$$v^2 \geq 2ad \quad \text{or,} \quad v \geq \sqrt{2ad}$$

Example 4. A man is standing 40 m behind the bus. Bus starts with 1 m/sec^2 constant acceleration and also at the same instant the man starts moving with constant speed 9 m/s. Find the time taken by man to catch the bus.



Solution : Let after time 't' man will catch the bus

For bus

$x = x_0 + ut + \frac{1}{2}at^2$, $\quad x = 40 + 0(t) + \frac{1}{2}(1)t^2$

$$x = 40 + \frac{t^2}{2} \qquad ....(i)$$

For man, $\quad x = 9t \qquad ....(ii)$

From (i) & (ii)

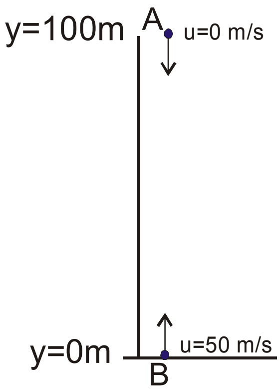
$$40 + \frac{t^2}{2} = 9t \quad \text{or} \quad t = 8 \text{ s} \quad \text{or} \quad t = 10\text{s}.$$

Example 5. A particle is dropped from height 100 m and another particle is projected vertically up with velocity 50 m/s from the ground along the same line. Find out the position where two particle will meet ? (take g = 10 m/s^2)

Solution : Let the upward direction is positive.

Let the particles meet at a distance y from the ground.

For particle A,



$y_0 = +100$ m
u = 0 m/s
a = – 10 m/s^2
$y = 100 + 0(t) - \frac{1}{2} 10 \times t^2 [y = y_0 + ut + \frac{1}{2}at^2]$
$= 100 - 5t^2$ .....(1)
For particle B,
$y_0 = 0$ m
u = + 50 m/s
a = – 10 m/s^2
$y = 50(t) - 10t^2$
$= 50t - 5t^2$ ......(2)
According to the problem;
$50t - 5t^2 = 100 - 5t^2$
t = 2 sec
Putting t = 2 sec in eqn. (1),
y = 100 – 20 = 80 m
Hence, the particles will meet at a height 80 m above the ground.

Example 6. A particle is dropped from a tower. It is found that it travels 45 m in the last second of its journey. Find out the height of the tower ? (take g = 10 m/s^2)

Solution : Let the total time of journey be n seconds.

Using; $s_n = u + \frac{a}{2}(2n-1)$ $\Rightarrow$ $45 = 0 + \frac{10}{2}(2n-1)$

n = 5 sec

Height of tower ; $\frac{1}{2}h = gt^2 = \frac{1}{2} \times 10 \times 5^2 = 125$ m

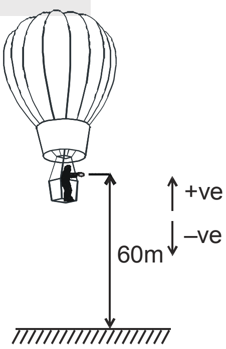
8. REACTION TIME

When a situation demands our immediate action. It takes some time before we really respond. Reaction time is the time a person takes to observe, think and act.

Solved Example

Example 1. A stone is dropped from a balloon going up with a uniform velocity of 5 m/s. If the balloon was 60 m high when the stone was dropped, find its height when the stone hits the ground. Take g = 10 m/s^2.

Solution : $S = ut + \frac{1}{2}at^2$



$-60 = 5(t) + \frac{1}{2}(-10)t^2$

$-60 = 5t - 5t^2$

$5t^2 - 5t - 60 = 0$

$t^2 - t - 12 = 0$

$t^2 - 4t + 3t - 12 = 0$

$(t-4)(t+3) = 0$

$\therefore$ t = 4

Height of balloon from ground at this instant

= 60 + 4 × 5 = 80 m

Example 2. A balloon is rising with constant acceleration 2 m/sec^2. Two stones are released from the balloon at the interval of 2 sec. Find out the distance between the two stones 1 sec. after the release of second stone.

Solution : Acceleration of balloon = 2 m/sec^2

Let at t = 0, y = 0 when the first stone is released.

By the question, $y_1 = 0\,t_1 + \frac{1}{2}gt_1^2$ (taking vertical upward as – ve and downward as + ve)

$\therefore$ Position of Ist stone = $\frac{9}{2}g$

(1 second after release of second stone will be the 3rd second for the 1st stone)

For second stone $y_2 = ut_2 + \frac{1}{2}gt_2^2$

u = 0 + at = – 2 × 2 = – 4m/s (taking vertical upward as – ve and downward as + ve)

$\therefore$ $y_2 = -4 \times 1 + \frac{1}{2}g \times (1)^2$ (t_2 = 1 second)

The 2nd stone is released after 2 second

$\therefore$ $y = -\frac{1}{2}at^2 = -\frac{1}{2} \times 2 \times 4 = -4$

So, Position of second stone from the origin = $-4 + \frac{1}{2}g - 4$

Distance between two stones = $\frac{1}{2}g \times 9 - \frac{1}{2}g \times 1 + 8 = 48$ m.

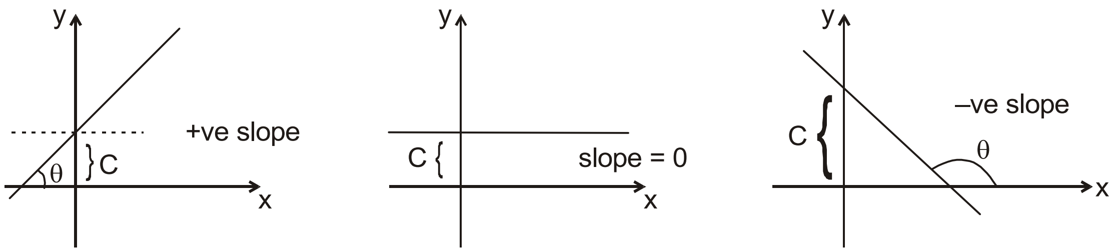
Note :
- As the particle is detached from the balloon it is having the same velocity as that of balloon, but its acceleration is only due to gravity and is equal to g.

9. STRAIGHT LINE-EQUATION, GRAPH, SLOPE (+VE, –VE, ZERO SLOPE).

If θ is the angle at which a straight line is inclined to the positive direction of x-axis, & $0° \le \theta < 180°$, $\theta \ne 90°$, then the slope of the line, denoted by m, is defined by m = tan θ. If θ is 90°, m does not exist, but the line is parallel to the y-axis. If θ = 0, then m = 0 & the line is parallel to the x-axis.

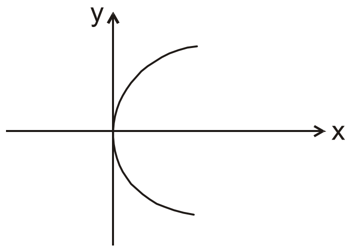
Slope – intercept form : y = mx + c is the equation of a straight line whose slope is m & which makes an intercept c on the y-axis.

$$m = \text{slope} = \tan\theta = \frac{dy}{dx}$$

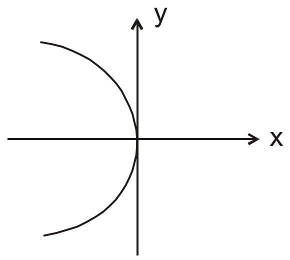


10. PARABOLIC CURVE-EQUATION, GRAPH (VARIOUS SITUATIONS UP, DOWN, LEFT, RIGHT WITH CONDITIONS)

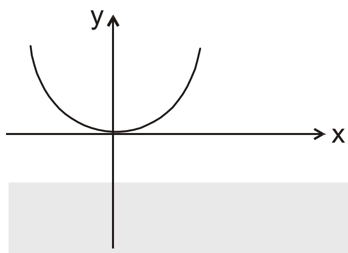
$y^2 = kx$



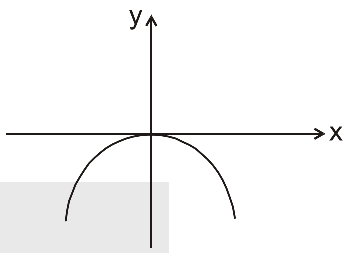
$y^2 = -kx$



$x^2 = ky$



$x^2 = -ky$



Where k is a positive constant.

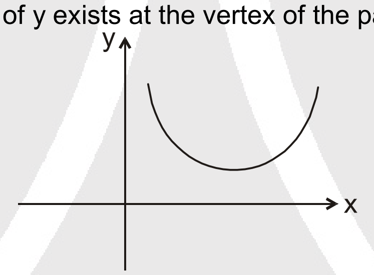
Equation of parabola :

Case (i) : $y = ax^2 + bx + c$

For $a > 0$

The nature of the parabola will be like that of the of nature $x^2 = ky$

Minimum value of y exists at the vertex of the parabola.

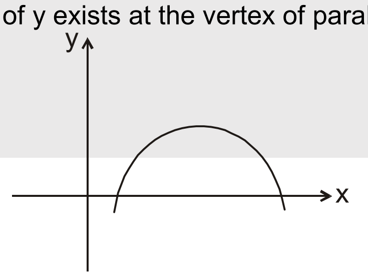


$y_{min} = \frac{-D}{4a}$ where $D = b^2 - 4ac$; Coordinates of vertex $= \left(\frac{-b}{2a}, \frac{D}{4a}\right)$

Case (ii) : $a < 0$

The nature of the parabola will be like that of the nature of $x^2 = -ky$

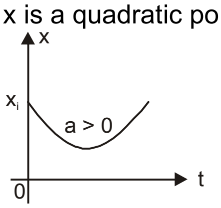
Maximum value of y exists at the vertex of parabola.

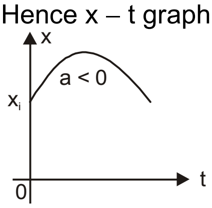


$y_{max} = D/4a$ where $D = b^2 - 4ac$

11. GRAPHS IN UNIFORMLY ACCELERATED MOTION ($A \neq 0$)

- x is a quadratic polynomial in terms of t. Hence x – t graph is a parabola.



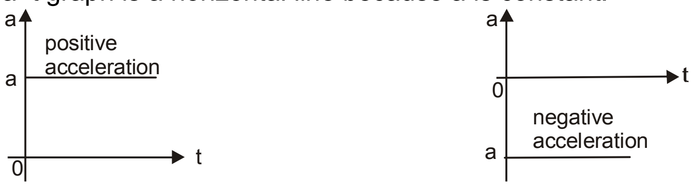


x-t graph

- v is a linear polynomial in terms of t. Hence v–t graph is a straight line of slope a.

v-t graph

- a–t graph is a horizontal line because a is constant.



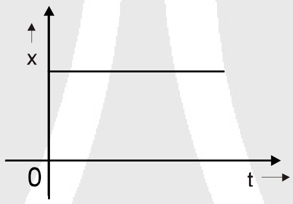
a-t graph

12. INTERPRETATION OF SOME MORE GRAPHS

12.1 Position vs Time graph

12.1.1 Zero Velocity

As position of particle is fixed at all the time, so the body is at rest.



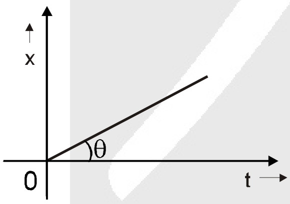
Slope; dx/dt = tanθ = tan 0º = 0

Velocity of particle is zero

12.1.2 Uniform Velocity

Here tan θ is constant tanθ = dx/dt

$\therefore$ dx/dt is constant.



$\therefore$ Velocity of particle is constant.

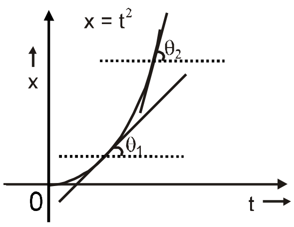
12.1.3 Non uniform velocity (increasing with time)

In this case;

As time is increasing, θ is also increasing.

$\therefore$ dx/dt = tanθ is also increasing

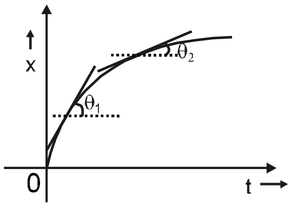
Hence, velocity of particle is increasing.



12.1.4 Non uniform velocity (decreasing with time)

In this case;

As time increases, θ decreases.



$\therefore \quad \frac{dx}{dt} = \tan\theta$ also decreases.

Hence, velocity of particle is decreasing.

12.2 Velocity vs time graph

12.2.1 Zero acceleration

Velocity is constant.

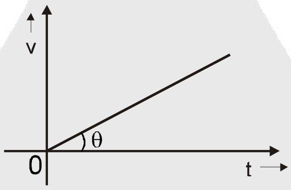
$\tan\theta = 0$

$\therefore \quad \frac{dv}{dt} = 0$

Hence, acceleration is zero.

12.2.2 Uniform acceleration

$\tan\theta$ is constant.



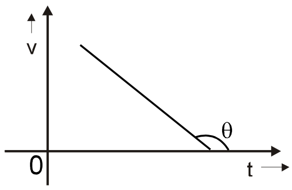
$\therefore \quad \frac{dv}{dt} = \text{constant}$

Hence, it shows constant acceleration.

12.2.3 Uniform retardation

Since $\theta > 90^o$

$\therefore \quad \tan\theta$ is constant and negative.

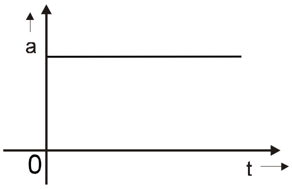


$\therefore \quad \frac{dv}{dt} = \text{negative constant}$

Hence, it shows constant retardation.

12.3 Acceleration vs time graph

12.3.1 Constant acceleration



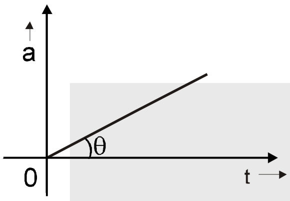
$\tan\theta = 0$

$\therefore$ da/dt = 0

Hence, acceleration is constant.

12.3.2 Uniformly increasing acceleration

θ is constant.



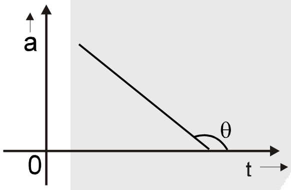
$0° < \theta < 90° \Rightarrow \tan\theta > 0$

$\therefore$ da/dt = $\tan\theta$ = constant > 0

Hence, acceleration is uniformly increasing with time.

12.3.3 Uniformly decreasing acceleration

Since $\theta > 90°$



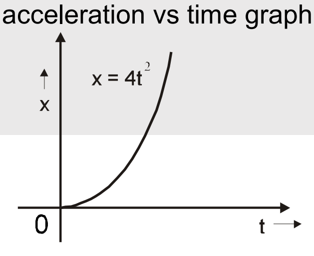
$\therefore$ $\tan\theta$ is constant and negative.

$\therefore$ da/dt = negative constant

Hence, acceleration is uniformly decreasing with time

Solved Example

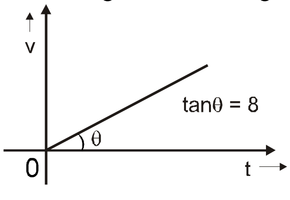
Example 1. The displacement vs time graph of a particle moving along a straight line is shown in the figure. Draw velocity vs time and acceleration vs time graph.



Solution : $x = 4t^2$

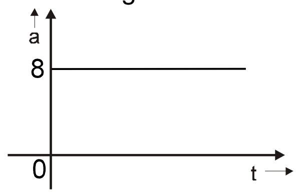
$v = dx/dt = 8t$

Hence, velocity-time graph is a straight line having slope i.e. $\tan\theta = 8$.

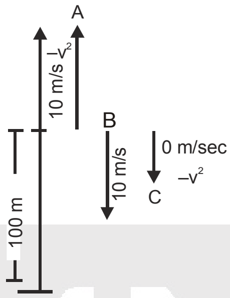


$$a = \frac{dv}{dt} = 8$$

Hence, acceleration is constant throughout and is equal to 8.



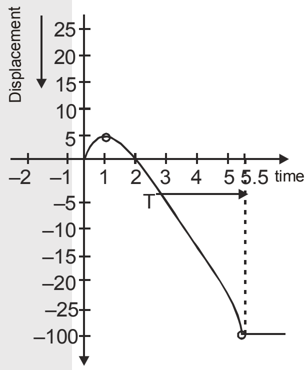
Example 2. At the height of 100 m, a particle A is thrown up with V = 10 m/s, B particle is thrown down with V = 10m/s and C particle released with V = 0 m/s. Draw graphs of each particle.



(i) Displacement–time (ii) Speed–time (iii) Velocity–time (iv) Acceleration–time

Solution : **For particle A :**

(i) Displacement vs time graph is



$y = ut + \frac{1}{2}at^2$

$u = + 10 \text{ m/sec}^2$

$y = 10t - \frac{1}{2} \times 10t^2 = 10\,t - 5t^2$

$v = \frac{dy}{dt} = 10 - 10\,t = 0$

t = 1 ; hence, velocity is zero at t = 1

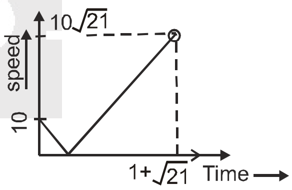
$10\,t - 5\,t^2 = -100$

$t^2 - 2t - 20 = 0$

t = 5.5 sec.

i.e., particle travels up till 5.5 seconds.

(ii) Speed vs time graph :



Particle has constant acceleration = g ↓ throughout the motion, so v-t curve will be straight line.

when moving up, v = u + at

0 = 10 – 10 t or t = 1 is the time at which speed is zero.

there after speed increases at constant rate of 10 m/s^2 .

Resulting Graph is : (speed is always positive).

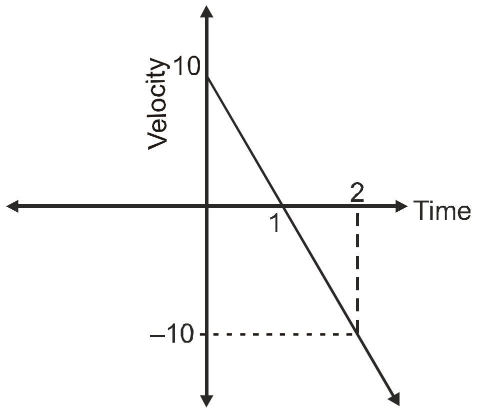
This shows that particle travels till a time of

$1 + \sqrt{21}$ seconds

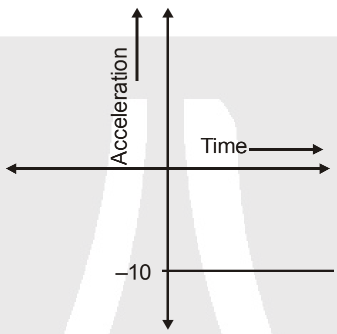
(iii) Velocity vs time graph :

V = u + at

V = 10 – 10t ; this shows that velocity becomes zero at t = 1 sec and thereafter the velocity is negative with slope g.

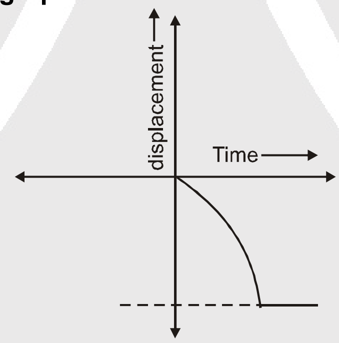


(iv) Acceleration vs time graph :
throughout the motion, particle has constant acceleration = -10 m/s^2.



For particle B : u = – 10 m/s. $y = -10t - \frac{1}{2}(10)t^2 = -10t - 5t^2$

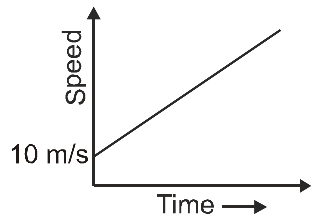
(i) Displacement time graph :



$y = 10t - 5t^2$; $\frac{dy}{dt} = -10t - 5t^2 = -10 - 10t$

this shows that slope becomes more negative with time.

(ii) Speed time graph :

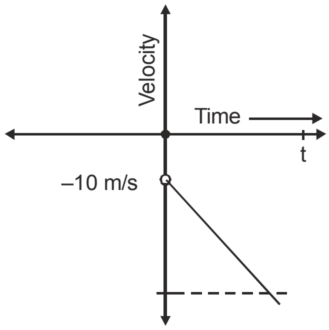


$\frac{dy}{dt} = -10t - 5t^2 = -10 - 10t$

hence, speed is directly proportional to time with slope of 10 initial speed = 10 m/s

(iii) Velocity time graph :

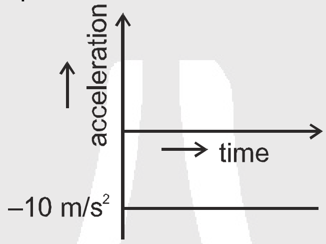
$\frac{dy}{dt} = -10t - 5t^2 = -10 - 10t$



hence, velocity is directly proportional to time with slope of –10. Initial velocity = –10 m/s

(iv) Acceleration vs time graph :

throughout the motion, particle has constant acceleration = $-10\ m/s^2$.

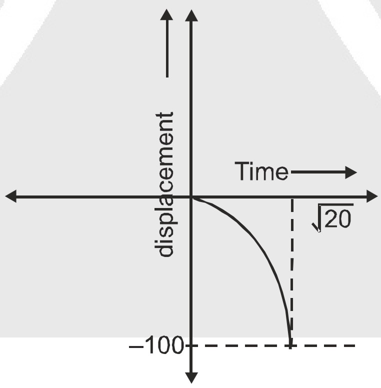


$a = \frac{dv}{dt} = -10$

For Particle C :

(i) Displacement time graph :

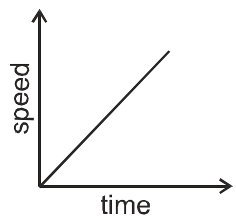
$u = 0,\ y = -\frac{1}{2} \times 10t^2 = -5t^2$



this shows that slope becomes more negative with time.

(ii) Speed vs time graph :

$v = \frac{dy}{dt} = -10\ t$

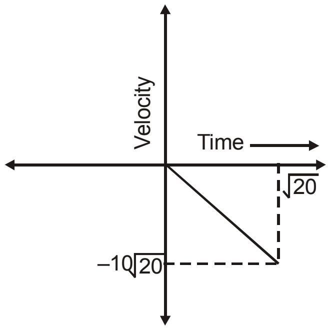


hence, speed is directly proportional to time with slope of 10.

(iii) Velocity time graph :

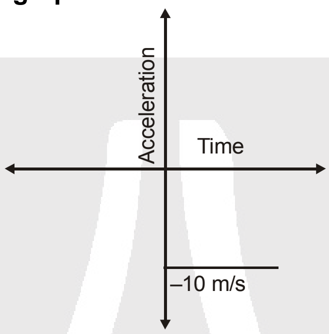
V = u + at

V = – 10t ;



hence, velocity is directly proportional to time with slope of –10.

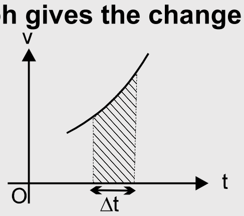
(iv) Acceleration vs time graph :



throughout the motion, particle has constant acceleration = –10 m/s^2.

13. DISPLACEMENT FROM V-T GRAPH & CHANGE IN VELOCITY FROM A -T GRAPH

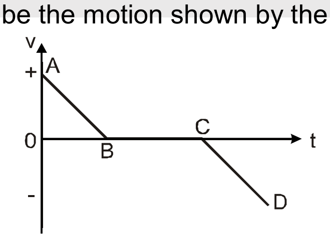
Displacement = Δx = area under v-t graph. Since a negative velocity causes a negative displacement, areas below the time axis are taken negative. In similar way, can see that $\Delta v = a\ \Delta t$ leads to the conclusion that **area under a – t graph gives the change in velocity Δv during that interval.**



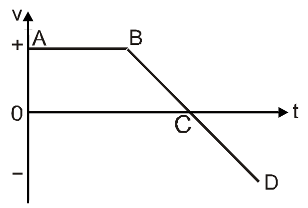
Solved Example

Example 1. Describe the motion shown by the following velocity-time graphs.

(a)



(b)



Solution :

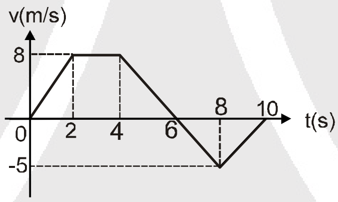
(a) **During interval AB:** velocity is +ve so the particle is moving in +ve direction, but it is slowing down as acceleration (slope of v-t curve) is negative. **During interval BC:** particle remains at rest as velocity is zero. Acceleration is also zero. **During interval CD:** velocity is -ve so the particle is moving in -ve direction and is speeding up as acceleration is also negative.

(b) **During interval AB:** particle is moving in +ve direction with constant velocity and acceleration is zero. **During interval BC:** particle is moving in +ve direction as velocity is +ve, but it slows down until it comes to rest as acceleration is negative. **During interval CD:** velocity is -ve so the particle is moving in -ve direction and is speeding up as acceleration is also negative.

Important Points to Remember

- For uniformly accelerated motion ($a \neq 0$), x–t graph is a parabola (opening upwards if $a > 0$ and opening downwards if $a < 0$). The slope of tangent at any point of the parabola gives the velocity at that instant.
- For uniformly accelerated motion ($a \neq 0$), v–t graph is a straight line whose slope gives the acceleration of the particle.
- In general, the slope of tangent in x–t graph is velocity and the slope of tangent in v–t graph is the acceleration.
- The area under a–t graph gives the change in velocity.
- The area between the v–t graph gives the distance travelled by the particle, if we take all areas as positive.
- Area under v–t graph gives displacement, if areas below the t–axis are taken negative.

Example 2. For a particle moving along x-axis, velocity-time graph is as shown in figure. Find the distance travelled and displacement of the particle?



Solution : Distance travelled = Area under v-t graph (taking all areas as +ve.)

Distance travelled = Area of trapezium + Area of triangle

$= \frac{1}{2}(2+6)\times 8 + \frac{1}{2}\times 4\times 5 = 32 + 10 = 42$ m

Displacement = Area under v-t graph (taking areas below time axis as –ve.)

Displacement = Area of trapezium – Area of triangle

$= \frac{1}{2}(2+6)\times 8 - \frac{1}{2}\times 4\times 5 = 32 - 10 = 22$ m

Hence, distance travelled = 42 m and displacement = 22 m.

14. MOTION WITH NON-UNIFORM ACCELERATION (USE OF DEFINITE INTEGRALS)

$\Delta x = \int_{t_i}^{t_f} v(t)dt$ (displacement in time interval $t = t_i$ to t_f)

The expression on the right hand side is called the definite integral of v(t) between $t = t_i$ and $t = t_f$. Similarly change in velocity

$\Delta v = v_f - v_i = \int_{t_i}^{t_f} a(t)dt$

14.1 Solving Problems which Involves Non uniform Acceleration

(i) Acceleration depending on velocity v or time t

By definition of acceleration, we have $a = \frac{dv}{dt}$. If a is in terms of t,

$\int_{v_0}^{v} dv = \int_0^t a(t)dt$. If a is in terms of v, $\int_{v_0}^{v} \frac{dv}{a(v)} = \int_0^t dt$.

On integrating, we get a relation between v and t, and then

using $\int_{x_0}^{x} dx = \int_0^t v(t)dt$, x and t can also be related.

(ii) Acceleration depending on velocity v or position x

$a = \frac{dv}{dt} \Rightarrow a = \frac{dv}{dx}\frac{dx}{dt} \Rightarrow a = \frac{dx}{dt}\frac{dv}{dx} \Rightarrow a = v\frac{dv}{dx}$

This is another important expression for acceleration. If a is in terms of x, $\int_{v_0}^{v} vdv = \int_{x_0}^{x} a(x)dx$.

If a is in terms of v,

On integrating, we get a relation between x and v.

Using $\int_{x_0}^{x} \frac{dx}{v(x)} = \int_0^t dt$, we can relate x and t.

Solved Example

Example 1. An object starts from rest at t = 0 and accelerates at a rate given by a = 6t. What is
(a) its velocity and
(b) its displacement at any time t?

Solution : As acceleration is given as a function of time, $\therefore \int_{v(t_0)}^{v(t)} dv = \int_{t_0}^{t} a(t)dt$

Here $t_0 = 0$ and $v(t_0) = 0$ $\quad \therefore \quad v(t) = \int_0^t 6tdt = 6\left(\frac{t^2}{2}\right)\Big|_0^t = 6\left(\frac{t^2}{2} - 0\right) = 3t^2$

So, $v(t) = 3t^2$

As $\Delta x = \int_{t_0}^{t} v(t)dt$ $\quad \therefore \quad \Delta x = \int_0^t 3t^2dt = 3\left(\frac{t^3}{3}\right)\Big|_0^t = 3\left(\frac{t^3}{3} - 0\right) = t^3$

Hence, velocity $v(t) = 3t^2$ and displacement $\Delta x = t^3$.

Example 2. For a particle moving along v + x-axis, acceleration is given as a = x. Find the position as a function of time? Given that at t = 0 , x = 1 v = 1.

Solution : $a = x \Rightarrow \frac{vdv}{dx} = x \Rightarrow \frac{v^2}{2} = \frac{x^2}{2} + C$

t = 0, x = 1 and v = 1

$\therefore C = 0 \Rightarrow v^2 = x^2$

$v = \pm x$ but given that x = 1 when v = 1

$\therefore \quad v = x \Rightarrow dx/dt = x \Rightarrow dx/x = dt$

$\ell nx = t + C \Rightarrow 0 = 0 + C \Rightarrow \ell nx = t \Rightarrow x = e^t$

Example 3. For a particle moving along x-axis, acceleration is given as a = v. Find the position as a function of time ? Given that at t = 0, x = 0 v = 1.

Solution : $a = v \Rightarrow \frac{dv}{dt} = v \Rightarrow \int \frac{dv}{v} = \int dt$

$\ell nv = t + c \Rightarrow 0 = 0 + c$

$v = e^t \Rightarrow \frac{dx}{dt} = e^t \Rightarrow \int dx = \int e^t dt$

$\Rightarrow x = e^t + c \Rightarrow 0 = 1 + c \Rightarrow x = e^t - 1$

MISCELLANEOUS SOLVED PROBLEMS

Problem 1 A particle covers 3/4 of total distance with speed v_1 and next 1/4 with v_2. Find the average speed of the particle?

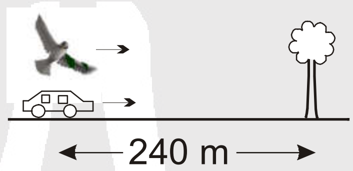
Answer : $\frac{4v_1v_2}{v_1 + 3v_2}$

Solution : Let the total distance be s

average speed (< v >) = $\frac{\text{Total distance}}{\text{Total time taken}}$

$$\langle v \rangle = \frac{s}{\frac{3s}{4v_1} + \frac{s}{4v_2}} = \frac{1}{\frac{3}{4v_1} + \frac{1}{4v_2}} = \frac{4v_1v_2}{v_1 + 3v_2}$$

Problem 2 A car is moving with speed 60 Km/h and a bird is moving with speed 90 km/h along the same direction as shown in figure. Find the distance travelled by the bird till the time car reaches the tree?



Answer : 360 m

Solution : Time taken by a car to reaches the tree (t) = $\frac{240 \text{ m}}{60 \text{ km/hr}} = \frac{0.24}{60}$hr

Now, the distance travelled by the bird during this time interval (s)

$= 90 \times \frac{0.24}{60}$ = 0.12 × 3 km = 360 m.

Problem 3 The position of a particle moving on X-axis is given by $x = At^3 + Bt^2 + Ct + D$. The numerical values of A, B, C, D are 1, 4, –2 and 5 respectively and SI units are used. Find (a) the dimensions of A, B, C and D, (b) the velocity of the particle at t = 4 s, (c) the acceleration of the particle at t = 4s, (d) the average velocity during the interval t =0 to t = 4s, (e) the average acceleration during the interval t = 0 to t = 4 s.

Answer : **(a)** $[A] = [LT^{-3}]$, $[B] = [LT^{-2}]$, $[C] = [LT^{-1}]$ and $[D] = [L]$; **(b)** 78 m/s ; **(c)** 32 m/s^2 ; **(d)** 30 m/s ; **(e)** 20 m/s^2

Solution : As $x = At^3 + Dt^2 + Ct + D$

(a) Dimensions of A, B, C and D,

$[At^3] = [x]$ (by principle of homogeneity)

$[A] = [LT^{-3}]$

Similarly, $[B] = [LT^{-2}]$, $[C] = [LT^{-1}]$ and $[D] = [L]$;

(b) As $v = dx/dt = 3At^2 + 2Bt + C$

Velocity at t = 4 sec.

$v = 3(1)(4)^2 + 2(4)(4) - 2 = 78$ m/s.

(c) Acceleration (a) = $dv/dt = 6At + 2B$; a = 32 m/s^2

(d) Average velocity as $x = At^3 + Bt^2 + Ct = D$

position at t = 0, is x = D = 5m.

Position at t = 4 sec is (1)(64) + (4)(16) – (2) (4) + 5 = 125 m

Thus the displacement during 0 to 4 sec. is 125 – 5 = 120 m

$\therefore$ < v > = 120 / 4 = 30 m/s

(e) $v = 3At^2 + 20t + C$, velocity at t = 0 is c = – 2 m/s

velocity at t = 4 sec is 78 m/s $\therefore \langle a \rangle = \frac{v_2 - v_1}{t_2 - t_1}$ = 20 m/s^2

Problem 4 For a particle moving along x-axis, velocity is given as a function of time as $v = 2t^2 + \sin t$. At $t = 0$, particle is at origin. Find the position as a function of time?

Solution : $v = 2t^2 + \sin t \Rightarrow dx/dt = 2t^2 + \sin t$

$$\int_0^x dx = \int_0^t (2t^2 + \sin t)dt = x = \frac{2}{3}t^3 - \cos t + 1$$ **Ans.**

Problem 5 A car decelerates from a speed of 20 m/s to rest in a distance of 100 m. What was its acceleration, assumed constant?

Solution : $v = 0$ $u = 20$ m/s $s = 100$ m $\Rightarrow$ as $v^2 = u^2 + 2as$

$0 = 400 + 2a \times 100 \Rightarrow a = -2$ m/s

$\therefore$ Acceleration = **2 m/s²** **Ans.**

Problem 6 A 150 m long train accelerates uniformly from rest. If the front of the train passes a railway worker 50 m away from the station at a speed of 25 m/s, what will be the speed of the back part of the train as it passes the worker?

Solution : $v^2 = u^2 + 2as$

$25 \times 25 = 0 + 100\,a$

$a = \frac{25}{4}$ m/s²

Now, for time taken by the back end of the train to pass the worker

we have $v'^2 = v^2 + 2al = (25)^2 + 2 \times 25/48 \times 150$

$v'^2 = 25 \times 25 \times 4$

$v' =$ **50 m/s.** **Ans.**

Problem 7 A particle is thrown vertically with velocity 20 m/s. Find (a) the distance travelled by the particle in first 3 seconds, (b) displacement of the particle in 3 seconds.

Answer : 25m, 15m

Solution : Highest point say B

$V_B = 0$

$v = u + gt$

$0 = 20 - 10\,t$

$t = 2$ sec.

$\therefore$ distance travel in first 2 seconds.

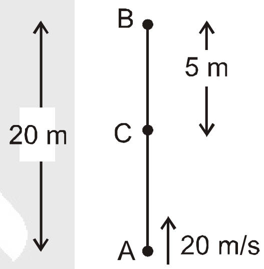
$s = s(t = 0 \text{ to } 2\text{sec}) + s(2\text{sec. to } 3\text{sec.})$

$s = [ut + 1/2\, at^2]_{t=0 \text{ to } t=2s} + [ut + 1/2 at^2]_{t=2 \text{ to } t=3s}$

$s = 20 \times 2 - 1/2 \times 10 \times 4 + 1/2 \times 10 \times 1^2$

$= (40 - 20) + 5 =$ **25 m.**

and displacement = 20 – 5 = **15 m.**



Problem 8 A car accelerates from rest at a constant rate α for some time after which it decelerates at a constant rate β to come to rest. If the total time elapsed is t. Find the maximum velocity acquired by the car.

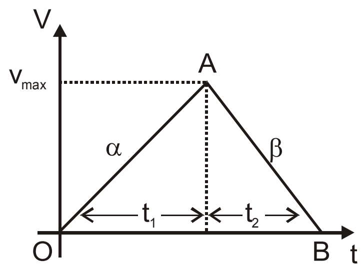
Solution : $t = t_1 + t_2$

slope of OA curve = $\tan\theta = \alpha = \frac{v_{max}}{t_1}$

slope of AB curve = $\beta = \frac{v_{max}}{t_2}$

$t = t_1 + t_2$

$\Rightarrow t = \frac{v_{max}}{\alpha} + \frac{v_{max}}{\beta} \Rightarrow v_{max} = \left(\frac{\alpha\beta}{\alpha+\beta}\right)t$



Problem 9 In the above question find total distance travelled by the car in time 't' .

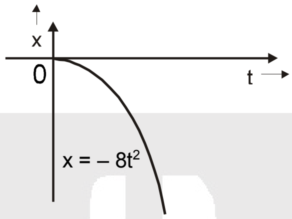
Solution : $v_{max} = \frac{\alpha\beta}{(\alpha+\beta)}t \Rightarrow t_1 = \frac{v_{max}}{\alpha} = \frac{\beta t}{(\alpha+\beta)} \Rightarrow t_2 = \frac{v_{max}}{\beta} = \frac{\alpha t}{(\alpha+\beta)}$

$\therefore$ Total distance travelled by the car in time 't'

$= \frac{1}{2}\alpha t_1^2 + v_{max}\, t_2 - \frac{1}{2}\beta t_2^2 = \frac{1}{2}\frac{\alpha\beta^2 t^2}{(\alpha+\beta)^2} + \frac{\alpha^2\beta t^2}{(\alpha+\beta)^2} - \frac{1}{2}\frac{\beta\alpha^2 t^2}{(\alpha+\beta)^2}$

Area under graph (directly) = $\frac{1}{2}\frac{\alpha\beta t^2}{(\alpha+\beta)} = \frac{\alpha\beta t^2}{2(\alpha+\beta)}$ **Ans.**

Problem 10 The displacement vs time graph of a particle moving along a straight line is shown in the figure. Draw velocity vs time and acceleration vs time graph.



Upwards direction is taken as positive, downwards direction is taken as negative.

Solution :

(a) The equation of motion is : $x = -8t^2$

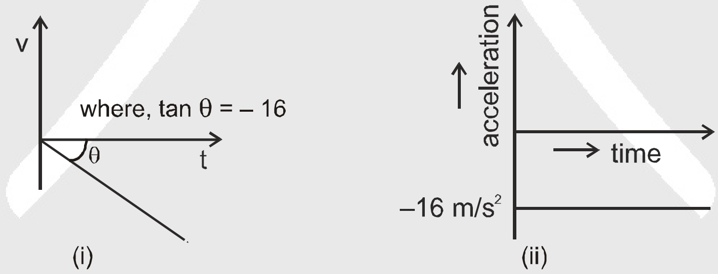
$\therefore$ $v = \frac{dx}{dt} = -16\,t$; this shows that velocity is directly proportional to time and slope of velocity-time curve is negative i.e., – 16.

Hence, resulting graph is (i)

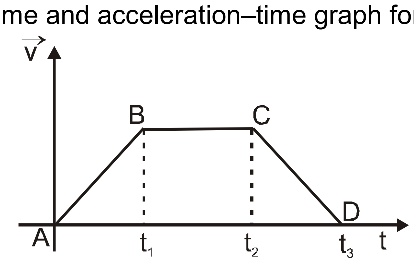
(b) Acceleration of particle is : $a = \frac{dv}{dt} = -16$.

This shows that acceleration is constant but negative.

Resulting graph is (ii)



Problem 11 Draw displacement–time and acceleration–time graph for the given velocity–time graph.



Solution : **Part AB :** v-t curve shows constant slope
i.e. constant acceleration or Velocity increases at constant rate with time.
Hence, s-t curve will show constant increase in slope
and a-t curve will be a straight line.

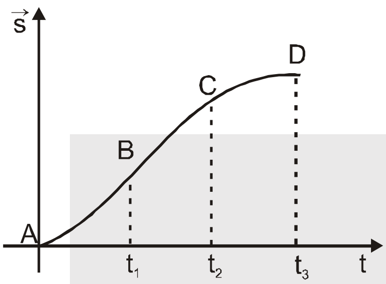
Part BC : v-t curve shows zero slope i.e. constant velocity. So, s-t curve will show constant slope and acceleration will be zero.

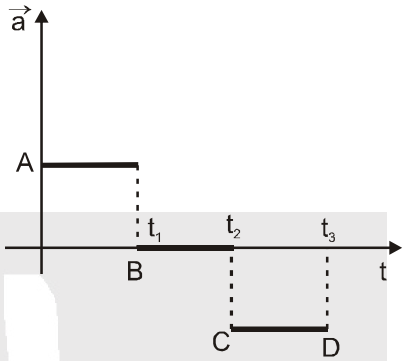
Part CD : v-t curve shows negative slope i.e. velocity is decreasing with time or acceleration is negative.

Hence, s-t curve will show decrease in slope becoming zero in the end.

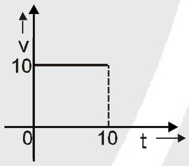
and a-t curve will be a straight line with negative intercept on y-axis.

RESULTING GRAPHS ARE :

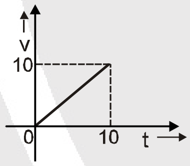




Problem 12 For a particle moving along x-axis, following graphs are given. Find the distance travelled by the particle in 10 s in each case?



(a)



(b)

Solution :

(a) Distance area under the v - t curve

$\therefore$ distance = $10 \times 10 = 100$ m **Ans.**

(b) Area under v – t curve

$\therefore$ distance = $\frac{1}{2} \times 10 \times 10 = 50$ m **Ans.**

Problem 13 For a particle moving along x-axis, acceleration is given as $a = 2v^2$. If the speed of the particle is v_0 at $x = 0$, find speed as a function of x.

Solution : $a = 2v^2 \Rightarrow$ or $\frac{dv}{dt} = 2v^2$ or $\frac{dv}{dx} \times \frac{dx}{dt} = 2v^2$

$v\frac{dv}{dx} = 2v^2 \Rightarrow \frac{dv}{dx} = 2v$

$\int_{v_0}^{v} \frac{dv}{v} = \int_0^x 2\ dx \Rightarrow [\ell nv]_{v_0}^{v} = [2x]_0^x$

$\ell n\frac{v}{v_0} = 2x \Rightarrow v = v_0e^{2x}$ **Ans.**