



Solution of Triangle

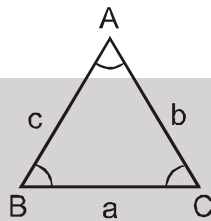
According to most accounts, geometry was first discovered among the Egyptians, taking its origin from the measurement of areas. For they found it necessary by reason of the flooding of the Nile, which wiped out everybody's proper boundaries. Nor is there anything surprising in that the discovery both of this and of the other sciences should have had its origin in a practical need, since everything which is in process of becoming progresses from the imperfect to the perfect.

.....Proclus

Sine Rule :

In any triangle ABC, the sines of the angles are proportional to the opposite sides

$$\text{i.e. } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$



Example # 1 : How many triangles can be constructed with the data : $a = 5, b = 7, \sin A = 3/4$

Solution : Since $\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{5}{3/4} = \frac{7}{\sin B}$
 $\Rightarrow \sin B = \frac{21}{20} > 1$ not possible
 \therefore no triangle can be constructed.

Example # 2 : If in a triangle ABC, $\frac{\sin A}{\sin C} = \frac{\sin(A - B)}{\sin(B - C)}$, then show that a^2, b^2, c^2 are in A.P.

Solution : We have $\frac{\sin A}{\sin C} = \frac{\sin(A - B)}{\sin(B - C)}$
 $\Rightarrow \sin(B + C) \sin(B - C) = \sin(A + B) \sin(A - B) \Rightarrow \sin^2 B - \sin^2 C = \sin^2 A - \sin^2 B$
 $\Rightarrow b^2 - c^2 = a^2 - b^2 \Rightarrow a^2, b^2, c^2$ are in A.P.

Self Practice Problems :

- (1) In a ΔABC , the sides a, b and c are in A.P., then prove that $\left(\tan \frac{A}{2} + \tan \frac{C}{2} \right) : \cot \frac{B}{2} = 2 : 3$
- (2) If the angles of ΔABC are in the ratio $1 : 2 : 3$, then find the ratio of their corresponding sides
- (3) In a ΔABC prove that $\frac{c}{a-b} = \frac{\tan \frac{A}{2} + \tan \frac{B}{2}}{\tan \frac{A}{2} - \tan \frac{B}{2}}$.

Ans. (2) $1 : \sqrt{3} : 2$

Cosine Formula :

In any ΔABC

(i) $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$ or $a^2 = b^2 + c^2 - 2bc \cos A = b^2 + c^2 + 2bc \cos(B + C)$

(ii) $\cos B = \frac{c^2 + a^2 - b^2}{2ca}$ (iii) $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$



Example # 3 : In a triangle ABC, A, B, C are in A.P. Show that $2\cos\left(\frac{A-C}{2}\right) = \frac{a+c}{\sqrt{a^2-ac+c^2}}$.

Solution : $A + C = 2B \Rightarrow A + B + C = 3B \Rightarrow B = 60^\circ$

$$\therefore \cos 60^\circ = \frac{a^2 + c^2 - b^2}{2ac} \Rightarrow a^2 - ac + c^2 = b^2$$

$$\begin{aligned} \Rightarrow \frac{a+c}{\sqrt{a^2-ac+c^2}} &= \frac{a+c}{b} = \left[\frac{\sin A + \sin C}{\sin B} \right] = \frac{2\sin\left(\frac{A+C}{2}\right)\cos\left(\frac{A-C}{2}\right)}{\sin B} \\ &= 2\cos\frac{A-C}{2} \quad (\because A+C=2B) \end{aligned}$$

Example # 4 : In a ΔABC , prove that $a(b\cos C - c\cos B) = b^2 - c^2$

Solution : Since $\cos C = \frac{a^2 + b^2 - c^2}{2ab}$ & $\cos B = \frac{a^2 + c^2 - b^2}{2ac}$

$$\begin{aligned} \therefore \text{L.H.S.} &= a \left\{ b \left(\frac{a^2 + b^2 - c^2}{2ab} \right) - c \left(\frac{a^2 + c^2 - b^2}{2ac} \right) \right\} \\ &= \frac{a^2 + b^2 - c^2}{2} - \frac{(a^2 + c^2 - b^2)}{2} = (b^2 - c^2) = \text{R.H.S.} \end{aligned}$$

Hence L.H.S. = R.H.S.

Proved

Example # 5 : The sides of ΔABC are $AB = \sqrt{13}$ cm, $BC = 4\sqrt{3}$ cm and $CA = 7$ cm. Then find the value of $\sin\theta$ where θ is the smallest angle of the triangle.

Solution : Angle opposite to AB is smallest. Therefore,

$$\cos\theta = \frac{49 + 48 - 13}{2 \cdot 7 \cdot 4\sqrt{3}} = \frac{\sqrt{3}}{2} \Rightarrow \sin\theta = \frac{1}{2}$$

Self Practice Problems :

(4) If in a triangle ABC, $3\sin A = 6\sin B = 2\sqrt{3}\sin C$, Then find the angle A.

(5) If two sides a, b and angle A be such that two triangles are formed, then find the sum of two values of the third side.

Ans. (4) 90° (5) $2b\cos A$

Projection Formula :

In any ΔABC

(i) $a = b\cos C + c\cos B$ (ii) $b = c\cos A + a\cos C$ (iii) $c = a\cos B + b\cos A$

Example # 6 : If in a ΔABC , $c\cos^2\frac{A}{2} + a\cos^2\frac{C}{2} = \frac{3b}{2}$, then show that a, b, c are in A.P.

Solution : $c(1 + \cos A) + a(1 + \cos C) = 3b$
 $\Rightarrow a + c + (c\cos A + a\cos C) = 3b$
 $\Rightarrow a + c + b = 3b$
 $\Rightarrow a + c = 2b$

Example # 7 : In a ΔABC , prove that $(b + c)\cos A + (c + a)\cos B + (a + b)\cos C = a + b + c$.

Solution : \therefore L.H.S. $= (b + c)\cos A + (c + a)\cos B + (a + b)\cos C$
 $= b\cos A + c\cos A + c\cos B + a\cos B + a\cos C + b\cos C$
 $= (b\cos A + a\cos B) + (c\cos A + a\cos C) + (c\cos B + b\cos C)$
 $= a + b + c$
 $= \text{R.H.S.}$

Hence L.H.S. = R.H.S.

Proved



Self Practice Problems :

- (6) The roots of $x^2 - 2\sqrt{3}x + 2 = 0$ represent two sides of a triangle. If the angle between them is $\frac{\pi}{3}$, then find the perimeter of triangle.
- (7) In a triangle ABC, if $\cos A + \cos B + \cos C = 3/2$, then show that the triangle is an equilateral triangle.
- (8) In a ΔABC , prove that $\frac{\cos A}{c \cos B + b \cos C} + \frac{\cos B}{a \cos C + c \cos A} + \frac{\cos C}{a \cos B + b \cos A} = \frac{a^2 + b^2 + c^2}{2abc}$.

Ans. (6) $2\sqrt{3} + \sqrt{6}$

Napier's Analogy - tangent rule :

In any ΔABC

(i) $\tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2}$ (ii) $\tan \frac{C-A}{2} = \frac{c-a}{c+a} \cot \frac{B}{2}$

(iii) $\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2}$

Example # 8 : Find the unknown elements of the ΔABC in which $a = \sqrt{3} + 1, b = \sqrt{3} - 1, C = 90^\circ$.

Solution :

$\therefore a = \sqrt{3} + 1, b = \sqrt{3} - 1, C = 90^\circ$

$\therefore A + B + C = 180^\circ$

$\therefore A + B = 90^\circ$ (i)

\therefore From law of tangent, we know that $\tan \left(\frac{A-B}{2} \right) = \frac{a-b}{a+b} \cot \frac{C}{2}$

$= \frac{(\sqrt{3}+1) - (\sqrt{3}-1)}{(\sqrt{3}+1) + (\sqrt{3}-1)} \cot 45^\circ = \frac{2}{2\sqrt{3}} \cot 45^\circ \Rightarrow \tan \left(\frac{A-B}{2} \right) = \frac{1}{\sqrt{3}}$

$\therefore \frac{A-B}{2} = \frac{\pi}{6}$

$\Rightarrow A - B = \frac{\pi}{3}$ (ii)

From equation (i) and (ii), we get $A = \frac{5\pi}{12}$ and $B = \frac{\pi}{12}$

Now, $c = \sqrt{a^2 + b^2} = 2\sqrt{2}$

$\therefore c = 2\sqrt{2}, A = \frac{5\pi}{12}, B = \frac{\pi}{12}$ **Ans.**

Self Practice Problems :

- (9) In a ΔABC if $b = 3, c = 5$ and $\cos (B - C) = \frac{7}{25}$, then find the value of $\sin \frac{A}{2}$.
- (10) If in a ΔABC , we define $x = \tan \left(\frac{B-C}{2} \right) \tan \frac{A}{2}, y = \tan \left(\frac{C-A}{2} \right) \tan \frac{B}{2}$ and $z = \tan \left(\frac{A-B}{2} \right) \tan \frac{C}{2}$, then show that $x + y + z = -xyz$.

Ans. (9) $\frac{1}{\sqrt{10}}$



Trigonometric Functions of Half Angles :

- (i) $\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$, $\sin \frac{B}{2} = \sqrt{\frac{(s-c)(s-a)}{ca}}$, $\sin \frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{ab}}$
- (ii) $\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$, $\cos \frac{B}{2} = \sqrt{\frac{s(s-b)}{ca}}$, $\cos \frac{C}{2} = \sqrt{\frac{s(s-c)}{ab}}$
- (iii) $\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{\Delta}{s(s-a)} = \frac{(s-b)(s-c)}{\Delta}$, where $s = \frac{a+b+c}{2}$ is semi perimeter and Δ is the area of triangle.
- (iv) $\sin A = \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)} = \frac{2\Delta}{bc}$

Area of Triangle (Δ)

$$\Delta = \frac{1}{2} ab \sin C = \frac{1}{2} bc \sin A = \frac{1}{2} ca \sin B = \sqrt{s(s-a)(s-b)(s-c)}$$

Example # 9 : If p_1, p_2, p_3 are the altitudes of a triangle ABC from the vertices A, B, C and Δ is the area of the triangle, then show that $p_1^{-1} + p_2^{-1} - p_3^{-1} = \frac{s-c}{\Delta}$

Solution : We have

$$\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} = \frac{a}{2\Delta} + \frac{b}{2\Delta} - \frac{c}{2\Delta}$$

$$= \frac{a+b-c}{2\Delta} = \frac{2(s-c)}{2\Delta} = \frac{s-c}{\Delta}$$

Example # 10 : In a ΔABC if $b \sin C(b \cos C + c \cos B) = 64$, then find the area of the ΔABC .

Solution : $\therefore b \sin C (b \cos C + c \cos B) = 64$ (i) given
 \therefore From **projection rule**, we know that
 $a = b \cos C + c \cos B$ put in (i), we get
 $ab \sin C = 64$ (ii)
 $\therefore \Delta = \frac{1}{2} ab \sin C$ \therefore from equation (ii), we get
 $\therefore \Delta = 32$ sq. unit

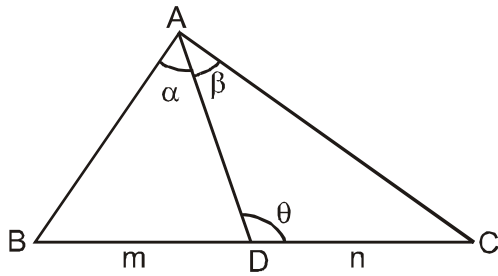
Example # 11 : If A,B,C are the angle of a triangle, then prove that $\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} = \frac{s^2}{\Delta}$

Solution : $\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2}$

$$= \sqrt{\frac{s(s-a)}{(s-b)(s-c)}} + \sqrt{\frac{s(s-b)}{(s-c)(s-a)}} + \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

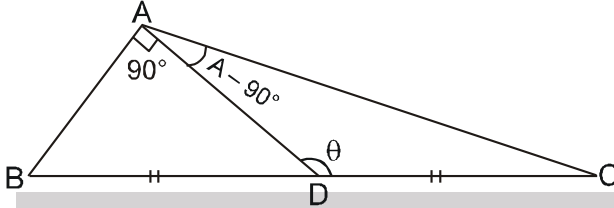
$$= \frac{\sqrt{s(s-a+s-b+s-c)}}{\sqrt{(s-a)(s-b)(s-c)}} = \frac{s}{\Delta} (3s - 2s) = \frac{s^2}{\Delta}$$

m - n Rule : In any triangle ABC if D be any point on the base BC, such that $BD : DC :: m : n$ and if $\angle BAD = \alpha$, $\angle DAC = \beta$, $\angle CDA = \theta$, then
 $(m+n) \cot \theta = m \cot \alpha - n \cot \beta$
 $n \cot B - m \cot C$



Example # 12 : In a ΔABC . AD divides BC in the ratio 2 : 1 such that at $\angle BAD = 90^\circ$ then prove that $\tan A + 3 \tan B = 0$

Solution : From the figure , we see that $\theta = 90^\circ + B$ (as θ is external angle of ΔABD)



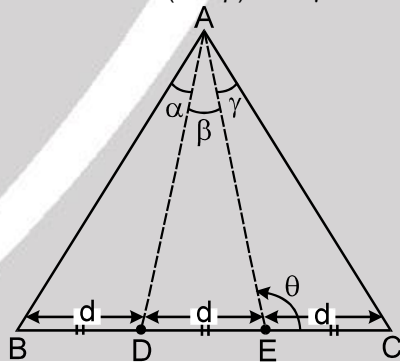
Now if we apply **m-n rule** in ΔABC , we get
 $(2 + 1) \cot (90^\circ + B) = 2 \cdot \cot 90^\circ - 1 \cdot \cot (A - 90^\circ)$
 $\Rightarrow -3 \tan B = \cot (90^\circ - A)$
 $\Rightarrow -3 \tan B = \tan A$
 $\Rightarrow \tan A + 3 \tan B = 0$ **Hence proved.**

Example # 13 : The base of a Δ is divided into three equal parts . If α, β, γ be the angles subtended by these parts at the vertex, prove that :

$$(\cot \alpha + \cot \beta) (\cot \beta + \cot \gamma) = 4 \operatorname{cosec}^2 \beta$$

Solution : Let point D and E divides the base BC into three equal parts i.e. $BD = DE = EC = d$ (Let) and let α, β and γ be the angles subtended by BD, DE and EC respectively at their opposite vertex. Now in ΔABC

$$\begin{aligned} \therefore BE : EC &= 2d : d = 2 : 1 \\ \therefore \text{from } \mathbf{m-n \text{ rule}}, \text{ we get} \\ (2 + 1) \cot \theta &= 2 \cot (\alpha + \beta) - \cot \gamma \\ \Rightarrow 3 \cot \theta &= 2 \cot (\alpha + \beta) - \cot \gamma \quad \dots\dots\dots(i) \end{aligned}$$



again

$$\begin{aligned} \therefore \text{in } \Delta ADC \\ \therefore DE : EC &= d : d = 1 : 1 \\ \therefore \text{if we apply } \mathbf{m-n \text{ rule}} \text{ in } \Delta ADC, \text{ we get} \\ (1 + 1) \cot \theta &= 1 \cdot \cot \beta - 1 \cdot \cot \gamma \\ 2 \cot \theta &= \cot \beta - \cot \gamma \quad \dots\dots\dots(ii) \end{aligned}$$

$$\text{from (i) and (ii), we get } \frac{3 \cot \theta}{2 \cot \theta} = \frac{2 \cot (\alpha + \beta) - \cot \gamma}{\cot \beta - \cot \gamma}$$

$$\Rightarrow 3 \cot \beta - 3 \cot \gamma = 4 \cot (\alpha + \beta) - 2 \cot \gamma$$



$$\begin{aligned} \Rightarrow 3\cot\beta - \cot\gamma &= 4 \cot(\alpha + \beta) \\ \Rightarrow 3\cot\beta - \cot\gamma &= 4 \left\{ \frac{\cot\alpha \cdot \cot\beta - 1}{\cot\beta + \cot\alpha} \right\} \\ \Rightarrow 3\cot^2\beta + 3\cot\alpha \cot\beta - \cot\beta \cot\gamma - \cot\alpha \cot\gamma &= 4 \cot\alpha \cot\beta - 4 \\ \Rightarrow 4 + 3\cot^2\beta &= \cot\alpha \cot\beta + \cot\beta \cot\gamma + \cot\alpha \cot\gamma \\ \Rightarrow 4 + 4\cot^2\beta &= \cot\alpha \cot\beta + \cot\alpha \cot\gamma + \cot\beta \cot\gamma + \cot^2\beta \\ \Rightarrow 4(1 + \cot^2\beta) &= (\cot\alpha + \cot\beta)(\cot\beta + \cot\gamma) \\ \Rightarrow (\cot\alpha + \cot\beta)(\cot\beta + \cot\gamma) &= 4\operatorname{cosec}^2\beta \end{aligned}$$

Self Practice Problems :

(11) In a ΔABC , the median to the side BC is of length $\frac{1}{\sqrt{11-6\sqrt{3}}}$ unit and it divides angle A into the angles of 30° and 45° . Prove that the side BC is of length 2 unit.

Radius of Circumcircle :

If R be the circumradius of ΔABC , then $R = \frac{a}{2\sin A} = \frac{b}{2\sin B} = \frac{c}{2\sin C} = \frac{abc}{4\Delta}$

Example # 14 : In a ΔABC , prove that $\sin 2A + \sin 2B + \sin 2C = 2\Delta/R^2$

Solution : In a ΔABC , we know that $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$
and $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$
 $= \frac{4abc}{8R^3} = \frac{16\Delta R}{8R^3} = \frac{2\Delta}{R^2}$

Example # 15 : In a ΔABC if $a = 22$ cm, $b = 28$ cm and $c = 36$ cm, then find its circumradius.

Solution : $\therefore R = \frac{abc}{4\Delta}$ (i)
 $\therefore \Delta = \sqrt{s(s-a)(s-b)(s-c)}$
 $\therefore s = \frac{a+b+c}{2} = 43$ cm
 $\therefore \Delta = \sqrt{43 \times 21 \times 15 \times 7} = 21\sqrt{215}$
 $\therefore R = \frac{22 \times 28 \times 36}{4 \times 21\sqrt{215}} = \frac{264}{\sqrt{215}}$ cm

Example # 16 : In a ΔABC , if $8R^2 = a^2 + b^2 + c^2$, show that the triangle is right angled.

Solution : We have : $8R^2 = a^2 + b^2 + c^2$
 $\Rightarrow 8R^2 = [4R^2 \sin^2 A + 4R^2 \sin^2 B + 4R^2 \sin^2 C]$ [$\because a = 2R \sin A$ etc.]
 $\Rightarrow 2 = \sin^2 A + \sin^2 B + \sin^2 C \Rightarrow (1 - \sin^2 A) - \sin^2 B + (1 - \sin^2 C) = 0$
 $\Rightarrow (\cos^2 A - \sin^2 B) + \cos^2 C = 0 \Rightarrow \cos(A+B) \cos(A-B) + \cos^2 C = 0$
 $\Rightarrow -\cos C \cos(A-B) + \cos^2 C = 0 \Rightarrow -\cos C \{\cos(A-B) - \cos C\} = 0$
 $\Rightarrow -\cos C [\cos(A-B) + \cos(A+B)] = 0 \Rightarrow -2\cos A \cos B \cos C = 0$
 $\Rightarrow \cos A = 0$ or $\cos B = 0$ or $\cos C = 0$
 $\Rightarrow A = \frac{\pi}{2}$ or $B = \frac{\pi}{2}$ or $C = \frac{\pi}{2}$
 $\Rightarrow \Delta ABC$ is a right angled triangle.

Example # 17 : $\frac{b^2 - c^2}{2a} = R \sin(B - C)$

Solution : $\frac{b^2 - c^2}{2a} = \frac{4R^2(\sin^2 B - \sin^2 C)}{4R \sin A} = \frac{R \sin(B+C) \sin(B-C)}{\sin A} = R \sin(B - C)$



Self Practice Problems :

(12) In a ΔABC , prove that $(a + b) = 4R \cos\left(\frac{A-B}{2}\right) \cos \frac{C}{2}$

(13) In a ΔABC , if $b = 15$ cm and $\cos B = \frac{4}{5}$, find R .

(14) In a triangle ABC if α, β, γ are the distances of the vertices of triangle from the corresponding points of contact with the incircle, then prove that $\frac{\alpha\beta\gamma}{\alpha + \beta + \gamma} = r^2$

Ans. (13) 12.5

Radius of The Incircle :

If 'r' be the inradius of ΔABC , then

(i) $r = \frac{\Delta}{s}$

(ii) $r = (s-a) \tan \frac{A}{2} = (s-b) \tan \frac{B}{2} = (s-c) \tan \frac{C}{2}$

(iii) $r = \frac{a \sin \frac{B}{2} \sin \frac{C}{2}}{\cos \frac{A}{2}}$ and so on

(iv) $r = 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

Radius of The Ex-Circles :

If r_1, r_2, r_3 are the radii of the ex-circles of ΔABC opposite to the vertex A, B, C respectively, then

(i) $r_1 = \frac{\Delta}{s-a}; r_2 = \frac{\Delta}{s-b}; r_3 = \frac{\Delta}{s-c};$

(ii) $r_1 = s \tan \frac{A}{2}; r_2 = s \tan \frac{B}{2}; r_3 = s \tan \frac{C}{2}$

(iii) $r_1 = \frac{a \cos \frac{B}{2} \cos \frac{C}{2}}{\cos \frac{A}{2}}$ and so on

(iv) $r_1 = 4R \sin \frac{A}{2} \cdot \cos \frac{B}{2} \cdot \cos \frac{C}{2}$

Example # 18 : $\cos A + \cos B + \cos C = \left(1 + \frac{r}{R}\right)$

Solution :

$$\begin{aligned} \text{LHS} &= \cos A + \cos B + \cos C \\ &= 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right) + 1 - 2 \sin^2 \frac{C}{2} \\ &= 2 \sin \frac{C}{2} \left\{ \cos\left(\frac{A-B}{2}\right) - \sin \frac{C}{2} \right\} + 1 = 2 \sin \frac{C}{2} \left\{ \cos\left(\frac{A-B}{2}\right) - \cos\left(\frac{A+B}{2}\right) \right\} + 1 \\ &= 2 \sin \frac{C}{2} \left\{ 2 \sin \frac{A}{2} \sin \frac{B}{2} \right\} + 1 = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \\ &= 1 + \frac{1}{R} \left(4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} \right) = 1 + \frac{r}{R} = \text{RHS} \end{aligned}$$

Example # 19 : In a triangle ABC , find the value of $\frac{b-c}{r_1} + \frac{c-a}{r_2} + \frac{a-b}{r_3}$.

Solution :

$$\begin{aligned} &\frac{b-c}{\left(\frac{\Delta}{s-a}\right)} + \frac{c-a}{\left(\frac{\Delta}{s-b}\right)} + \frac{a-b}{\left(\frac{\Delta}{s-c}\right)} \\ &= \frac{1}{\Delta} [(b-c)(s-a) + (c-a)(s-b) + (a-b)(s-c)] \end{aligned}$$



$$= \frac{1}{\Delta} [s(b-c+c-a+a-b) - a(b-c) - b(c-a) - c(a-b)] = 0$$

Self Practice Problems :

- (15) In a triangle ABC, r_1, r_2, r_3 are in HP. If its area is 24 cm^2 and its perimeter is 24 cm . then find lengths of its sides.
- (16) In a triangle ABC, $a : b : c = 4 : 5 : 6$. Find the ratio of the radius of the circumcircle to that of the incircle.
- (17) In a ΔABC , prove that $\frac{r_1-r}{a} + \frac{r_2-r}{b} = \frac{c}{r_3}$.
- (18) If A, A_1, A_2 and A_3 are the areas of the inscribed and escribed circles respectively of a ΔABC , then prove that $\frac{1}{\sqrt{A}} = \frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} + \frac{1}{\sqrt{A_3}}$.

Ans. (15) 6, 8, 10 (16) 16 : 7

Length of Angle Bisectors, Medians & Altitudes :

- (i) Length of an angle bisector from the angle $A = \beta_a = \frac{2bc \cos \frac{A}{2}}{b+c}$;
- (ii) Length of median from the angle $A = m_a = \frac{1}{2} \sqrt{2b^2 + 2c^2 - a^2}$
- & (iii) Length of altitude from the angle $A = A_a = \frac{2\Delta}{a}$

NOTE : $m_a^2 + m_b^2 + m_c^2 = \frac{3}{4} (a^2 + b^2 + c^2)$

Example # 20 : In ΔABC , AD & BE are its two median . If $AD = 4$, $\angle DAB = \frac{\pi}{6}$ and $\angle ABE = \frac{\pi}{3}$ then find the length of BE and area of ΔABC .

Solution : $AP = \frac{2}{3}$; $AD = \frac{8}{3}$; $PD = \frac{4}{3}$; Let $PB = x$

$$\tan 60^\circ = \frac{8/3}{x} \quad \text{or} \quad x = \frac{8}{3\sqrt{3}}$$

$$\text{Area of } \Delta ABP = \frac{1}{2} \times \frac{8}{3} \times \frac{8}{3\sqrt{3}} = \frac{32}{9\sqrt{3}}$$

$$\therefore \text{Area of } \Delta ABC = 3 \times \frac{32}{9\sqrt{3}} = \frac{32}{3\sqrt{3}}$$

$$\text{Also, } BE = \frac{3}{2}x = \frac{4}{\sqrt{3}}$$

Self Practice Problem :

- (19) In a ΔABC if $\angle A = 90^\circ$, $b = 5 \text{ cm}$, $c = 12 \text{ cm}$. If 'G' is the centroid of triangle, then find circumradius of ΔGAB .



Ans. (19) $\frac{13\sqrt{601}}{30}$ cm

The Distances of The Special Points from Vertices and Sides of Triangle :

- (i) Circumcentre (O) : $OA = R$ and $O_a = R \cos A$
- (ii) Incentre (I) : $IA = r \operatorname{cosec} \frac{A}{2}$ and $I_a = r$
- (iii) Excentre (I_1) : $I_1 A = r_1 \operatorname{cosec} \frac{A}{2}$ and $I_{1a} = r_1$
- (iv) Orthocentre (H) : $HA = 2R \cos A$ and $H_a = 2R \cos B \cos C$
- (v) Centroid (G) : $GA = \frac{1}{3} \sqrt{2b^2 + 2c^2 - a^2}$ and $G_a = \frac{2\Delta}{3a}$

Example # 21 : If p_1, p_2, p_3 are respectively the lengths of perpendiculars from the vertices of a triangle ABC to the opposite sides, prove that :

(i) $\frac{\cos A}{p_1} + \frac{\cos B}{p_2} + \frac{\cos C}{p_3} = \frac{1}{R}$ (ii) $\frac{bp_1}{c} + \frac{cp_2}{a} + \frac{ap_3}{b} = \frac{a^2 + b^2 + c^2}{2R}$

Solution :

(i) use $\frac{1}{p_1} = \frac{a}{2\Delta}, \frac{1}{p_2} = \frac{b}{2\Delta}, \frac{1}{p_3} = \frac{c}{2\Delta}$

$$\begin{aligned} \therefore \text{LHS} &= \frac{1}{2\Delta} (a \cos A + b \cos B + c \cos C) \\ &= \frac{R}{2\Delta} (\sin 2A + \sin 2B + \sin 2C) = \frac{4R \sin A \sin B \sin C}{2\Delta} \\ &= \frac{4R}{2\Delta} \cdot \frac{a}{2R} \cdot \frac{b}{2R} \cdot \frac{c}{2R} = \frac{1}{4\Delta R^2} abc = \frac{1}{4\Delta R^2} \cdot (4R\Delta) = \frac{1}{R} = \text{RHS} \end{aligned}$$

(ii)
$$\begin{aligned} \text{LHS} &= \frac{bp_1}{c} + \frac{cp_2}{a} + \frac{ap_3}{b} = \frac{a^2 + b^2 + c^2}{2R} = \frac{2b\Delta}{ac} + \frac{2c\Delta}{ab} + \frac{2a\Delta}{bc} = \frac{2\Delta(a^2 + b^2 + c^2)}{abc} \\ &= \frac{2\Delta(a^2 + b^2 + c^2)}{4\Delta R} = \frac{a^2 + b^2 + c^2}{2R} \end{aligned}$$

Self Practice Problems :

(20) If I be the incentre of ΔABC , then prove that $IA \cdot IB \cdot IC = abc \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}$.

(21) If x, y, z are respectively be the perpendiculars from the circumcentre to the sides of ΔABC ,

then prove that $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{4xyz}$.



Exercise-1

Marked questions are recommended for Revision.

SUBJECTIVE QUESTIONS

Section (A) : Sine rule, Cosine rule, Napier's Analogy, Projection rule

A-1. In a $\triangle ABC$, prove that :

- (i) $a \sin(B - C) + b \sin(C - A) + c \sin(A - B) = 0$
- (ii) $\frac{a^2 \sin(B - C)}{\sin A} + \frac{b^2 \sin(C - A)}{\sin B} + \frac{c^2 \sin(A - B)}{\sin C} = 0$
- (iii) $2(bc \cos A + ca \cos B + ab \cos C) = a^2 + b^2 + c^2$
- (iv) $(a - b)^2 \cos^2 \frac{C}{2} + (a + b)^2 \sin^2 \frac{C}{2} = c^2$
- (v) $b^2 \sin 2C + c^2 \sin 2B = 2bc \sin A$
- (vi) $\frac{\sin B}{\sin C} = \frac{c - a \cos B}{b - a \cos C}$

A-2. Find the real value of x such that $x^2 + 2x$, $2x + 3$ and $x^2 + 3x + 8$ are lengths of the sides of a triangle.

A-3. The angles of a $\triangle ABC$ are in A.P. (order being A, B, C) and it is being given that $b : c = \sqrt{3} : \sqrt{2}$, then find $\angle A$.

A-4. If $\cos A + \cos B = 4 \sin^2 \left(\frac{C}{2} \right)$, prove that sides a, c, b of the triangle ABC are in A.P.

A-5. If in a $\triangle ABC$, $\frac{\sin A}{\sin C} = \frac{\sin(A - B)}{\sin(B - C)}$, then prove that a^2, b^2, c^2 are in A.P.

A-6. In a triangle ABC , prove that for any angle θ , $b \cos(A - \theta) + a \cos(B + \theta) = c \cos \theta$.

A-7. With usual notations, if in a $\triangle ABC$, $\frac{b + c}{11} = \frac{c + a}{12} = \frac{a + b}{13}$, then prove that $\frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$.

A-8. Let a, b and c be the sides of a $\triangle ABC$. If a^2, b^2 and c^2 are the roots of the equation $x^3 - Px^2 + Qx - R = 0$, where P, Q & R are constants, then find the value of $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$ in terms of P, Q and R .

A-9. If in a triangle ABC , the altitude AM be the bisector of $\angle BAD$, where D is the mid point of side BC , then prove that $(b^2 - c^2) = a^2/2$.

A-10. If in a triangle ABC , $\angle C = 60^\circ$, then prove that $\frac{1}{a+c} + \frac{1}{b+c} = \frac{3}{a+b+c}$

A-11. In a triangle ABC , $\angle C = 60^\circ$ and $\angle A = 75^\circ$. If D is a point on AC such that the area of the $\triangle ABD$ is $\sqrt{3}$ times the area of the $\triangle BCD$, find the $\angle ABD$.

A-12. In a scalene triangle ABC , D is a point on the side AB such that $CD^2 = AD \cdot DB$, if $\sin A \cdot \sin B = \sin^2 \frac{C}{2}$ then prove that CD is internal bisector of $\angle C$.

A-13. In triangle ABC , D is on AC such that $AD = BC$, $BD = DC$, $\angle DBC = 2x$, and $\angle BAD = 3x$, all angles are in degrees, then find the value of x .



Section (B) Trigonometric ratios of Half Angles, Area of triangle and circumradius

B-1. In a $\triangle ABC$, prove that

$$(i) \quad 2 \left[a \sin^2 \frac{C}{2} + c \sin^2 \frac{A}{2} \right] = c + a - b.$$

$$(ii) \quad \frac{\cos^2 \frac{A}{2}}{a} + \frac{\cos^2 \frac{B}{2}}{b} + \frac{\cos^2 \frac{C}{2}}{c} = \frac{s^2}{abc}$$

$$(iii) \quad 4 \left(bc \cos^2 \frac{A}{2} + ca \cos^2 \frac{B}{2} + ab \cos^2 \frac{C}{2} \right) = (a + b + c)^2$$

$$(iv) \quad (b - c) \cot \frac{A}{2} + (c - a) \cot \frac{B}{2} + (a - b) \cot \frac{C}{2} = 0$$

$$(v) \quad 4\Delta (\cot A + \cot B + \cot C) = a^2 + b^2 + c^2$$

$$(vi) \quad \left(\frac{2abc}{a+b+c} \right) \cdot \cos \frac{A}{2} \cdot \cos \frac{B}{2} \cdot \cos \frac{C}{2} = \Delta$$

B-2. If the sides a, b, c of a triangle are in A.P., then find the value of $\tan \frac{A}{2} + \tan \frac{C}{2}$ in terms of $\cot (B/2)$.

B-3. If in a $\triangle ABC$, $a = 6, b = 3$ and $\cos(A - B) = 4/5$, then find its area.

B-4. If in a triangle ABC , $\angle A = 30^\circ$ and the area of triangle is $\frac{\sqrt{3} a^2}{4}$, then prove that either $B = 4C$ or $C = 4B$.

Section (C) Inradius and Exradius

C-1. In any $\triangle ABC$, prove that

$$(i) \quad Rr (\sin A + \sin B + \sin C) = \Delta$$

$$(ii) \quad a \cos B \cos C + b \cos C \cos A + c \cos A \cos B = \frac{\Delta}{R}$$

$$(iii) \quad \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = \frac{1}{2Rr}.$$

$$(iv) \quad \cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} = 2 + \frac{r}{2R}$$

$$(v) \quad a \cot A + b \cot B + c \cot C = 2(R + r)$$

C-2. In any $\triangle ABC$, prove that

$$(i) \quad r_1 \cdot r_2 \cdot r_3 = \Delta^2$$

$$(ii) \quad r_1 + r_2 - r_3 + r = 4R \cos C.$$

$$(iii) \quad \frac{1}{r^2} + \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} = \frac{a^2 + b^2 + c^2}{\Delta^2}$$

$$(iv) \quad \left(\frac{1}{r} + \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \right)^2 = \frac{4}{r} \left(\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \right)$$

$$(v) \quad \frac{bc - r_2 r_3}{r_1} = \frac{ca - r_3 r_1}{r_2} = \frac{ab - r_1 r_2}{r_3} = r$$



- C-3.** Show that the radii of the three escribed circles of a triangle are roots of the equation $x^3 - x^2(4R + r) + x s^2 - r s^2 = 0$.
- C-4.** The radii r_1, r_2, r_3 of escribed circles of a triangle ABC are in harmonic progression. If its area is 24 sq. cm and its perimeter is 24 cm, find the lengths of its sides.
- C-5.** If the area of a triangle is 100 sq.cm, $r_1 = 10$ cm and $r_2 = 50$ cm, then find the value of $(b - a)$.

Section (D) Miscellaneous

- D-1.** If α, β, γ are the respective altitudes of a triangle ABC, prove that
- (i)
$$\frac{1}{\alpha^2} + \frac{1}{\beta^2} + \frac{1}{\gamma^2} = \frac{\cot A + \cot B + \cot C}{\Delta}$$
- (ii)
$$\frac{1}{\alpha} + \frac{1}{\beta} - \frac{1}{\gamma} = \frac{2ab}{(a + b + c) \Delta} \cos^2 \frac{C}{2}$$
- D-2.** If in an acute angled ΔABC , line joining the circumcentre and orthocentre is parallel to side AC, then find the value of $\tan A \cdot \tan C$.
- D-3.** A regular hexagon & a regular dodecagon are inscribed in the same circle. If the side of the dodecagon is $(\sqrt{3} - 1)$, if the side of the hexagon is $\sqrt[4]{k}$, then find value of k.
- D-4.** If D is the mid point of CA in triangle ABC and Δ is the area of triangle, then show that
$$\tan(\angle ADB) = \frac{4\Delta}{a^2 - c^2}$$
.

Exercise-2

Marked questions are recommended for Revision.

PART-I (OBJECTIVE QUESTIONS)

Section (A) : Sine rule, Cosine rule, Napier's Analogy, Projection rule

- A-1.** In a ΔABC , $A : B : C = 3 : 5 : 4$. Then $a + b + c \sqrt{2}$ is equal to
 (A) $2b$ (B) $2c$ (C) $3b$ (D) $3a$
- A-2*.** In a triangle ABC, the altitude from A is not less than BC and the altitude from B is not less than AC. The triangle is
 (A) right angled (B) isosceles (C) obtuse angled (D) equilateral
- A-3.** If in a ΔABC , $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$, then the triangle is :
 (A) right angled (B) isosceles (C) equilateral (D) obtuse angled
- A-4.** In a ΔABC $\frac{bc \sin^2 A}{\cos A + \cos B \cos C}$ is equal to
 (A) $b^2 + c^2$ (B) bc (C) a^2 (D) $a^2 + bc$
- A-5.** Given a triangle ΔABC such that $\sin^2 A + \sin^2 C = 1001 \cdot \sin^2 B$. Then the value of $\frac{2(\tan A + \tan C) \cdot \tan^2 B}{\tan A + \tan B + \tan C}$ is
 (A) $\frac{1}{2000}$ (B) $\frac{1}{1000}$ (C) $\frac{1}{500}$ (D) $\frac{1}{250}$



- A-6.** If in a triangle ABC, $(a + b + c)(b + c - a) = k \cdot bc$, then :
 (A) $k < 0$ (B) $k > 6$ (C) $0 < k < 4$ (D) $k > 4$
- A-7.** In a triangle ABC, $a : b : c = 4 : 5 : 6$. Then $3A + B$ equals to :
 (A) $4C$ (B) 2π (C) $\pi - C$ (D) π
- A-8.** The distance between the middle point of BC and the foot of the perpendicular from A is ($b > c$) :
 (A) $\frac{-a^2 + b^2 + c^2}{2a}$ (B) $\frac{b^2 - c^2}{2a}$ (C) $\frac{b^2 + c^2}{\sqrt{bc}}$ (D) $\frac{b^2 + c^2}{2a}$
- A-9*.** If in a triangle ABC, $\cos A \cos B + \sin A \sin B \sin C = 1$, then the triangle is
 (A) isosceles (B) right angled (C) equilateral (D) None of these
- A-10.** Triangle ABC is right angle at A. The points P and Q are on hypotenuse BC such that $BP = PQ = QC$. If $AP = 3$ and $AQ = 4$, then length BC is equal to
 (A) $3\sqrt{5}$ (B) $5\sqrt{3}$ (C) $4\sqrt{5}$ (D) 7
- A-11.** In ΔABC , $bc = 2b^2 \cos A + 2c^2 \cos A - 4bc \cos^2 A$, then ΔABC is
 (A) isosceles but not necessarily equilateral
 (B) equilateral
 (C) right angled but not necessarily isosceles
 (D) right angled isosceles

Section (B) Trigonometric ratios of Half Angles, Area of triangle and circumradius

- B-1.** If in a triangle ABC, right angle at B, $s - a = 3$ and $s - c = 2$, then
 (A) $a = 2, c = 3$ (B) $a = 3, c = 4$ (C) $a = 4, c = 3$ (D) $a = 6, c = 8$
- B-2.** If in a triangle ABC, $b \cos^2 \frac{A}{2} + a \cos^2 \frac{B}{2} = \frac{3}{2}c$, then a, c, b are :
 (A) in A.P. (B) in G.P. (C) in H.P. (D) None
- B-3.** If H is the orthocentre of a triangle ABC, then the radii of the circle circumscribing the triangles BHC, CHA and AHB are respectively equal to :
 (A) R, R, R (B) $\sqrt{2}R, \sqrt{2}R, \sqrt{2}R$ (C) $2R, 2R, 2R$ (D) $\frac{R}{2}, \frac{R}{2}, \frac{R}{2}$
- B-4.** In a ΔABC if $b + c = 3a$, then $\cot \frac{B}{2} \cdot \cot \frac{C}{2}$ has the value equal to:
 (A) 4 (B) 3 (C) 2 (D) 1
- B-5.** In a ΔABC , $A = \frac{2\pi}{3}$, $b - c = 3\sqrt{3}$ cm and area (ΔABC) = $\frac{9\sqrt{3}}{2}$ cm². Then 'a' is
 (A) $6\sqrt{3}$ cm (B) 9 cm (C) 18 cm (D) 7 cm
- B-6.*** The diagonals of a parallelogram are inclined to each other at an angle of 45° , while its sides a and b ($a > b$) are inclined to each other at an angle of 30° , then the value of $\frac{a}{b}$ is
 (A) $2\cos 36^\circ$ (B) $\sqrt{\frac{3+\sqrt{5}}{4}}$ (C) $\frac{3+\sqrt{5}}{4}$ (D) $\frac{\sqrt{5}+1}{2}$



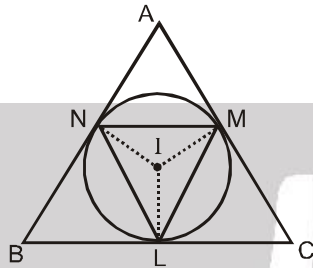
- B-7.** If in a ΔABC , $\Delta = a^2 - (b - c)^2$, then $\tan A$ is equal to
 (A) 15/16 (B) 8/15 (C) 8/17 (D) 1/2
- B-8*.** If in a ΔABC , $a = 5$, $b = 4$ and $\cos(A - B) = \frac{31}{32}$, then
 (A) $c = 6$ (B) $\sin A = \left(\frac{5\sqrt{7}}{16}\right)$
 (C) area of $\Delta ABC = \frac{15\sqrt{7}}{4}$ (D) $c = 8$
- B-9.** If R denotes circumradius, then in ΔABC , $\frac{b^2 - c^2}{2aR}$ is equal to
 (A) $\cos(B - C)$ (B) $\sin(B - C)$ (C) $\cos B - \cos C$ (D) $\sin(B + C)$
- B-10*.** Which of the following holds good for any triangle ABC ?
 (A) $\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c} = \frac{a^2 + b^2 + c^2}{2abc}$ (B) $\frac{\sin A}{a} + \frac{\sin B}{b} + \frac{\sin C}{c} = \frac{3}{2R}$
 (C) $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$ (D) $\frac{\sin 2A}{a^2} = \frac{\sin 2B}{b^2} = \frac{\sin 2C}{c^2}$
- B-11.** A triangle is inscribed in a circle. The vertices of the triangle divide the circle into three arcs of length 3, 4 and 5 units. Then area of the triangle is equal to:
 (A) $\frac{9\sqrt{3}(1+\sqrt{3})}{\pi^2}$ (B) $\frac{9\sqrt{3}(\sqrt{3}-1)}{\pi^2}$ (C) $\frac{9\sqrt{3}(1+\sqrt{3})}{2\pi^2}$ (D) $\frac{9\sqrt{3}(\sqrt{3}-1)}{2\pi^2}$
- B-12.** In a ΔABC , $a = 1$ and the perimeter is six times the arithmetic mean of the sines of the angles. Then measure of $\angle A$ is
 (A) $\frac{\pi}{3}$ (B) $\frac{\pi}{2}$ (C) $\frac{\pi}{6}$ (D) $\frac{\pi}{4}$
- B-13*.** Three equal circles of radius unity touches one another. Radius of the circle touching all the three circles is :
 (A) $\frac{2-\sqrt{3}}{\sqrt{3}}$ (B) $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{2}}$ (C) $\frac{2+\sqrt{3}}{\sqrt{3}}$ (D) $\frac{\sqrt{3}+\sqrt{2}}{\sqrt{2}}$
- B-14.** Triangle ABC is isosceles with $AB = AC$ and $BC = 65$ cm. P is a point on BC such that the perpendicular distances from P to AB and AC are 24 cm and 36 cm, respectively. The area of triangle ABC (in sq. cm) is
 (A) 1254 (B) 1950 (C) 2535 (D) 5070

Section (C) Inradius and Exradius

- C-1.** In a ΔABC , the value of $\frac{a\cos A + b\cos B + c\cos C}{a+b+c}$ is equal to:
 (A) $\frac{r}{R}$ (B) $\frac{R}{2r}$ (C) $\frac{R}{r}$ (D) $\frac{2r}{R}$
- C-2.** In a triangle ABC , if $a : b : c = 3 : 7 : 8$, then $R : r$ is equal to
 (A) 2 : 7 (B) 7 : 2 (C) 3 : 7 (D) 7 : 3
- C-3*.** If $r_1 = 2r_2 = 3r_3$, then
 (A) $\frac{a}{b} = \frac{4}{5}$ (B) $\frac{a}{b} = \frac{5}{4}$ (C) $\frac{a}{c} = \frac{3}{5}$ (D) $\frac{a}{c} = \frac{5}{3}$



- C-4*** In a ΔABC , following relations hold good. In which case(s) the triangle is a right angled triangle?
 (A) $r_2 + r_3 = r_1 - r$ (B) $a^2 + b^2 + c^2 = 8R^2$ (C) $r_1 = s$ (D) $2R = r_1 - r$
- C-5.** The perimeter of a triangle ABC right angled at C is 70, and the inradius is 6, then $|a - b|$ equals
 (A) 1 (B) 2 (C) 8 (D) 9
- C-6.** In a triangle ABC , if $\frac{a-b}{b-c} = \frac{s-a}{s-c}$, then r_1, r_2, r_3 are in:
 (A) A.P. (B) G.P. (C) H.P. (D) none of these
- C-7.** If the incircle of the ΔABC touches its sides at L, M and N as shown in the figure and if x, y, z be the circumradii of the triangles MIN, NIL and LIM respectively, where I is the incentre, then the product xyz is equal to :



- (A) Rr^2 (B) rR^2 (C) $\frac{1}{2}Rr^2$ (D) $\frac{1}{2}rR^2$
- C-8.** If in a ΔABC , $\frac{r}{r_1} = \frac{1}{2}$, then the value of $\tan \frac{A}{2} \left(\tan \frac{B}{2} + \tan \frac{C}{2} \right)$ is equal to :
 (A) 2 (B) $\frac{1}{2}$ (C) 1 (D) 3
- C-9.** If in a ΔABC , $\angle A = \frac{\pi}{2}$, then $\tan \frac{C}{2}$ is equal to
 (A) $\frac{a-c}{2b}$ (B) $\frac{a-b}{2c}$ (C) $\frac{a-c}{b}$ (D) $\frac{a-b}{c}$
- C-10.** In any ΔABC , $\frac{(r_1 + r_2)(r_2 + r_3)(r_3 + r_1)}{Rs^2}$ is always equal to
 (A) 8 (B) 27 (C) 16 (D) 4
- C-11*.** In a triangle ABC , right angled at B , then
 (A) $r = \frac{AB + BC - AC}{2}$ (B) $r = \frac{AB + AC - BC}{2}$
 (C) $r = \frac{AB + BC + AC}{2}$ (D) $R = \frac{s-r}{2}$
- C-12*.** With usual notations, in a ΔABC the value of $\prod (r_1 - r)$ can be simplified as:
 (A) $abc \prod \tan \frac{A}{2}$ (B) $4rR^2$ (C) $\frac{(abc)^2}{R(a+b+c)^2}$ (D) $4Rr^2$

- C-13. STATEMENT-1 :** In a triangle ABC , the harmonic mean of the three exradii is three times the inradius.
STATEMENT-2 : In any triangle ABC , $r_1 + r_2 + r_3 = 4R$.
- (A) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is correct explanation for STATEMENT-1
 (B) STATEMENT-1 is true, STATEMENT-2 is true and STATEMENT-2 is not correct explanation for STATEMENT-1
 (C) STATEMENT-1 is true, STATEMENT-2 is false
 (D) STATEMENT-1 is false, STATEMENT-2 is true
 (E) Both STATEMENTS are false



Section (D) Miscellaneous

- D-1.** If in a triangle ABC, the line joining the circumcentre and incentre is parallel to BC, then $\cos B + \cos C$ is equal to :
 (A) 0 (B) 1 (C) 2 (D) 1/2
- D-2.** In a ΔABC , if $AB = 5$ cm, $BC = 13$ cm and $CA = 12$ cm, then the distance of vertex 'A' from the side BC is (in cm)
 (A) $\frac{25}{13}$ (B) $\frac{60}{13}$ (C) $\frac{65}{12}$ (D) $\frac{144}{13}$
- D-3.** If AD, BE and CF are the medians of a ΔABC , then $(AD^2 + BE^2 + CF^2) : (BC^2 + CA^2 + AB^2)$ is equal to
 (A) 4 : 3 (B) 3 : 2 (C) 3 : 4 (D) 2 : 3
- D-4*.** In a triangle ABC, with usual notations the length of the bisector of internal angle A is :
 (A) $\frac{2bc \cos \frac{A}{2}}{b+c}$ (B) $\frac{2bc \sin \frac{A}{2}}{b+c}$ (C) $\frac{abc \operatorname{cosec} \frac{A}{2}}{2R(b+c)}$ (D) $\frac{2\Delta}{b+c} \cdot \operatorname{cosec} \frac{A}{2}$
- D-5.** Let f, g, h be the lengths of the perpendiculars from the circumcentre of the ΔABC on the sides BC, CA and AB respectively. If $\frac{a}{f} + \frac{b}{g} + \frac{c}{h} = \lambda \frac{abc}{fgh}$, then the value of ' λ ' is:
 (A) 1/4 (B) 1/2 (C) 1 (D) 2
- D-6.** In an acute angled triangle ABC, AP is the altitude. Circle drawn with AP as its diameter cuts the sides AB and AC at D and E respectively, then length DE is equal to
 (A) $\frac{\Delta}{2R}$ (B) $\frac{\Delta}{3R}$ (C) $\frac{\Delta}{4R}$ (D) $\frac{\Delta}{R}$
- D-7.** AA_1, BB_1 and CC_1 are the medians of triangle ABC whose centroid is G. If points A, C_1 , G and B_1 are concyclic, then
 (A) $2b^2 = a^2 + c^2$ (B) $2c^2 = a^2 + b^2$ (C) $2a^2 = b^2 + c^2$ (D) $3a^2 = b^2 + c^2$
- D-8.** If ' l ' is the length of median from the vertex A to the side BC of a ΔABC , then
 (A) $4l^2 = b^2 + 4ac \cos B$ (B) $4l^2 = a^2 + 4bc \cos A$
 (C) $4l^2 = c^2 + 4ab \cos C$ (D) $4l^2 = b^2 + 2c^2 - 2a^2$
- D-9*.** The product of the distances of the incentre from the angular points of a ΔABC is:
 (A) $4R^2r$ (B) $4Rr^2$ (C) $\frac{(abc)R}{s}$ (D) $\frac{(abc)r}{s}$
- D-10.** In a triangle ABC, $B = 60^\circ$ and $C = 45^\circ$. Let D divides BC internally in the ratio 1 : 3, then value of $\frac{\sin \angle BAD}{\sin \angle CAD}$ is
 (A) $\sqrt{\frac{2}{3}}$ (B) $\frac{1}{\sqrt{3}}$ (C) $\frac{1}{\sqrt{6}}$ (D) $\frac{1}{3}$
- D-11*.** In a triangle ABC, points D and E are taken on side BC such that $BD = DE = EC$. If angle $ADE = \text{angle } AED = \theta$, then:
 (A) $\tan \theta = 3 \tan B$ (B) $3 \tan \theta = \tan C$
 (C) $\frac{6 \tan \theta}{\tan^2 \theta - 9} = \tan A$ (D) angle B = angle C



7. Sides of the $\Delta I_1 I_2 I_3$ are

(A) $R \cos \frac{A}{2}, R \cos \frac{B}{2}, R \cos \frac{C}{2}$

(B) $4R \cos \frac{A}{2}, 4R \cos \frac{B}{2}, 4R \cos \frac{C}{2}$

(C) $2R \cos \frac{A}{2}, 2R \cos \frac{B}{2}, 2R \cos \frac{C}{2}$

(D) None of these

8. Value of $II_1^2 + I_2 I_3^2 = II_2^2 + I_3 I_1^2 = II_3^2 + I_1 I_2^2 =$

(A) $4R^2$

(B) $16R^2$

(C) $32R^2$

(D) $64R^2$

PART-III (MATCH THE COLUMN)

1. Match the column

Column-I

Column-II

(A) In a ΔABC , $2B = A + C$ and $b^2 = ac$.

(p) 8

Then the value of $\frac{a^2(a+b+c)}{3abc}$ is equal to

(B) In any right angled triangle ABC, the value of $\frac{a^2 + b^2 + c^2}{R^2}$ is always equal to (where R is the circumradius of ΔABC)

(q) 1

(C) In a ΔABC if $a = 2, bc = 9$, then the value of $2R\Delta$ is equal to

(r) 5

(D) In a ΔABC , $a = 5, b = 3$ and $c = 7$, then the value of $3 \cos C + 7 \cos B$ is equal to

(s) 9

2. Match the column

Column - I

Column - II

(A) In a ΔABC , $a = 4, b = 3$ and the medians AA_1 and BB_1 are mutually perpendicular, then square of area of the ΔABC is equal to

(p) 27

(B) In any ΔABC , minimum value of $\frac{r_1 r_2 r_3}{r^3}$ is equal to

(q) 7

(C) In a ΔABC , $a = 5, b = 4$ and $\tan \frac{C}{2} = \sqrt{\frac{7}{9}}$, then side 'c' is equal to

(r) 6

(D) In a ΔABC , $2a^2 + 4b^2 + c^2 = 4ab + 2ac$, then value of $(8 \cos B)$ is equal to

(s) 11

Exercise-3

Marked Questions may have for Revision Questions.

* Marked Questions may have more than one correct option.

PART - I : JEE (ADVANCED) / IIT-JEE PROBLEMS (PREVIOUS YEARS)

1. If the angle A, B and C of a triangle are in arithmetic progression and if a, b and c denote the lengths of the sides opposite to A, B and C respectively, then the value of the expression $\frac{a}{c} \sin 2C + \frac{c}{a} \sin 2A$ is

[IIT-JEE 2010, Paper-1, (3, -1), 84]

(A) $\frac{1}{2}$

(B) $\frac{\sqrt{3}}{2}$

(C) 1

(D) $\sqrt{3}$



2. Let ABC be a triangle such that $\angle ACB = \frac{\pi}{6}$ and let a, b and c denote the lengths of the sides opposite to A, B and C respectively. The value(s) of x for which $a = x^2 + x + 1$, $b = x^2 - 1$ and $c = 2x + 1$ is (are)
[IIT-JEE 2010, Paper-1, (3, 0), 84]
 (A) $-(2 + \sqrt{3})$ (B) $1 + \sqrt{3}$ (C) $2 + \sqrt{3}$ (D) $4\sqrt{3}$
3. Consider a triangle ABC and let a, b and c denote the lengths of the sides opposite to vertices A, B and C respectively. Suppose $a = 6$, $b = 10$ and the area of the triangle is $15\sqrt{3}$. If $\angle ACB$ is obtuse and if r denotes the radius of the incircle of the triangle, then r^2 is equal to
[IIT-JEE 2010, Paper-2, (3, 0), 79]
4. Let PQR be a triangle of area Δ with $a = 2$, $b = \frac{7}{2}$ and $c = \frac{5}{2}$, where a, b and c are the lengths of the sides of the triangle opposite to the angles at P, Q and R respectively. Then $\frac{2\sin P - \sin 2P}{2\sin P + \sin 2P}$ equals
[IIT-JEE 2012, Paper-2, (3, -1), 66]
 (A) $\frac{3}{4\Delta}$ (B) $\frac{45}{4\Delta}$ (C) $\left(\frac{3}{4\Delta}\right)^2$ (D) $\left(\frac{45}{4\Delta}\right)^2$
5. In a triangle PQR, P is the largest angle and $\cos P = \frac{1}{3}$. Further the incircle of the triangle touches the sides PQ, QR and RP at N, L and M respectively, such that the lengths of PN, QL and RM are consecutive even integers. Then possible length(s) of the side(s) of the triangle is (are)
[JEE (Advanced) 2013, Paper-2, (3, -1)/60]
 (A) 16 (B) 18 (C) 24 (D) 22
6. In a triangle the sum of two sides is x and the product of the same two sides is y. If $x^2 - c^2 = y$, where c is the third side of the triangle, then the ratio of the in-radius to the circum-radius of the triangle is
[JEE (Advanced) 2014, Paper-2, (3, -1)/60]
 (A) $\frac{3y}{2x(x+c)}$ (B) $\frac{3y}{2c(x+c)}$ (C) $\frac{3y}{4x(x+c)}$ (D) $\frac{3y}{4c(x+c)}$
7. In a triangle XYZ, let x, y, z be the lengths of sides opposite to the angles X, Y, Z, respectively, and $2s = x + y + z$. If $\frac{s-x}{4} = \frac{s-y}{3} = \frac{s-z}{2}$ and area of incircle of the triangle XYZ is $\frac{8\pi}{3}$, then
[JEE (Advanced) 2016, Paper-1, (4, -2)/62]
 (A) area of the triangle XYZ is $6\sqrt{6}$
 (B) the radius of circumcircle of the triangle XYZ is $\frac{35}{6}\sqrt{6}$
 (C) $\sin \frac{X}{2} \sin \frac{Y}{2} \sin \frac{Z}{2} = \frac{4}{35}$
 (D) $\sin^2 \left(\frac{X+Y}{2} \right) = \frac{3}{5}$
8. In a triangle PQR, let $\angle PQR = 30^\circ$ and the sides PQ and QR have lengths $10\sqrt{3}$ and 10, respectively. Then, which of the following statement(s) is (are) TRUE?
[JEE(Advanced) 2018, Paper-1, (4, -2)/60]
 (A) $\angle QPR = 45^\circ$
 (B) The area of the triangle PQR is $25\sqrt{3}$ and $\angle QRP = 120^\circ$
 (C) The radius of the incircle of the triangle PQR is $10\sqrt{3} - 15$
 (D) The area of the circumcircle of the triangle PQR is 100π



9. In a non-right-angled triangle ΔPQR , Let p, q, r denote the lengths of the sides opposite to the angles at P, Q, R respectively. The median from R meets the side PQ at S , the perpendicular from P meets the side QR at E , and RS and PE intersect at O . If $p = \sqrt{3}$, $q = 1$, and the radius of the circumcircle of the ΔPQR equals 1, then which of the following options is/are correct ?

[JEE(Advanced) 2019, Paper-1, (4, -1)/62]

- (A) Length of $RS = \frac{\sqrt{7}}{2}$ (B) Area of $\Delta SOE = \frac{\sqrt{3}}{12}$
 (C) Radius of incircle of $\Delta PQR = \frac{\sqrt{3}}{2}(2 - \sqrt{3})$ (D) Length of $OE = \frac{1}{6}$

PART - II : JEE (MAIN) / AIEEE PROBLEMS (PREVIOUS YEARS)

1. For a regular polygon, let r and R be the radii of the inscribed and the circumscribed circles. A false statement among the following is [AIEEE - 2010 (4, -1), 144]

- (1) There is a regular polygon with $\frac{r}{R} = \frac{1}{\sqrt{2}}$. (2) There is a regular polygon with $\frac{r}{R} = \frac{2}{3}$.
 (3) There is a regular polygon with $\frac{r}{R} = \frac{\sqrt{3}}{2}$. (4) There is a regular polygon with $\frac{r}{R} = \frac{1}{2}$.

2. ABCD is a trapezium such that AB and CD are parallel and $BC \perp CD$. If $\angle ADB = \theta$, $BC = p$ and $CD = q$, then AB is equal to : [AIEEE - 2013, (4, -1), 120]

- (1) $\frac{(p^2 + q^2) \sin \theta}{p \cos \theta + q \sin \theta}$ (2) $\frac{p^2 + q^2 \cos \theta}{p \cos \theta + q \sin \theta}$ (3) $\frac{p^2 + q^2}{p^2 \cos \theta + q^2 \sin \theta}$ (4) $\frac{(p^2 + q^2) \sin \theta}{(p \cos \theta + q \sin \theta)^2}$

3. With the usual notation, in ΔABC , if $\angle A + \angle B = 120^\circ$, $a = \sqrt{3} + 1$ and $b = \sqrt{3} - 1$, then the ratio $\angle A : \angle B$, is: [JEE(Main) 2019, Online (10-01-19), P-2 (4, -1), 120]

- (1) 9 : 7 (2) 7 : 1 (3) 3 : 1 (4) 5 : 3

4. In a triangle, the sum of lengths of two sides is x and the product of the lengths of the same two sides is y . If $x^2 - c^2 = y$, where c is the length of the third side of the triangle, then the circumradius of the triangle is [JEE(Main) 2019, Online (11-01-19), P-1 (4, -1), 120]

- (1) $\frac{c}{\sqrt{3}}$ (2) $\frac{3}{2}y$ (3) $\frac{c}{3}$ (4) $\frac{y}{\sqrt{3}}$



Answers

EXERCISE - 1

Section (A) :

A-2. $x > 5$ A-3. 75° A-8. $\frac{P}{2\sqrt{R}}$ A-11. 30° A-13. 10°

Section (B) :

B-2. $\frac{2}{3} \cot \frac{B}{2}$ B-3. 9 sq. unit

Section (C)

C-4. 6, 8, 10 cm C-5. 8

Section (D)

D-2. 3 D-3. $\sqrt{2}$

EXERCISE - 2

PART - I

Section (A) :

A-1. (C) A-2. (AB) A-3. (C) A-4. (C) A-5. (D) A-6. (C) A-7. (D)
 A-8. (B) A-9. (AB) A-10. (A) A-11. (A)

Section (B) :

B-1. (B) B-2. (A) B-3. (A) B-4. (C) B-5. (B) B-6. (AD) B-7. (B)
 B-8. (ABC) B-9. (B) B-10. (AB) B-11. (A) B-12. (C) B-13. (AC) B-14. (C)

Section (C) :

C-1. (A) C-2. (B) C-3. (BD) C-4. (ABCD) C-5. (A) C-6. (A) C-7. (C)
 C-8. (B) C-9. (D) C-10. (D) C-11*. (AD) C-12*. (ACD) C-13. (C)

Section (D)

D-1. (B) D-2. (B) D-3. (C) D-4. (ACD) D-5. (A) D-6. (D) D-7. (C)
 D-8. (B) D-9. (BD) D-10. (C) D-11. (ACD) D-12. (A)

PART - II

1. (A) 2. (BC) 3. (A) 4. (ABCD) 5. (B) 6. (A) 7. (B)
 8. (B)

PART - III

1. (A) → (q), (B) → (p), (C) → (s), (D) → (r)
 2. (A) → (s), (B) → (p), (C) → (r), (D) → (q)

EXERCISE - 3

PART - I

1. (D) 2. (B) 3. 3 4. (C) 5. (BD) 6. (B)
 7. (A,C,D) 8. (BCD) 9. (ACD)

PART - II

1. (2) 2. (1) 3. (2) 4. (1)



HLP Answers

SUBJECTIVE QUESTIONS

This questions paste Staright Line sheets

- In $\triangle ABC$, P is an interior point such that $\angle PAB = 10^\circ$, $\angle PBA = 20^\circ$, $\angle PCA = 30^\circ$, $\angle PAC = 40^\circ$ then prove that $\triangle ABC$ is isosceles
- In a triangle ABC, if $a \tan A + b \tan B = (a + b) \tan\left(\frac{A+B}{2}\right)$, prove that triangle is isosceles.
- In any triangle ABC, if $2\Delta a - b^2c = c^3$, (where Δ is the area of triangle), then prove that $\angle A$ is obtuse
- If in a triangle ABC, $\frac{\cos A + 2\cos C}{\cos A + 2\cos B} = \frac{\sin B}{\sin C}$ prove that the triangle ABC is either isosceles or right angled.
- In a $\triangle ABC$, $\angle C = 60^\circ$ and $\angle A = 75^\circ$. If D is a point on AC such that the area of the $\triangle BAD$ is $\sqrt{3}$ times the area of the $\triangle BCD$, find the $\angle ABD$.
- In a $\triangle ABC$, if a, b and c are in A.P., prove that $\cos A \cot \frac{A}{2}$, $\cos B \cot \frac{B}{2}$, and $\cos C \cot \frac{C}{2}$ are in A.P.
- In a triangle ABC, prove that the area of the incircle is to the area of triangle itself is,
 $\pi : \cot\left(\frac{A}{2}\right) \cdot \cot\left(\frac{B}{2}\right) \cdot \cot\left(\frac{C}{2}\right)$.
- In $\triangle ABC$, prove that $a^2(s-a) + b^2(s-b) + c^2(s-c) = 4R\Delta \left(1 + 4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$
- In any $\triangle ABC$, prove that
 - $(r_3 + r_1)(r_3 + r_2) \sin C = 2r_3 \sqrt{r_2 r_3 + r_3 r_1 + r_1 r_2}$
 - $\frac{\tan \frac{A}{2}}{(a-b)(a-c)} + \frac{\tan \frac{B}{2}}{(b-a)(b-c)} + \frac{\tan \frac{C}{2}}{(c-a)(c-b)} = \frac{1}{\Delta}$
 - $(r+r_1) \tan \frac{B-C}{2} + (r+r_2) \tan \frac{C-A}{2} + (r+r_3) \tan \frac{A-B}{2} = 0$
 - $r^2 + r_1^2 + r_2^2 + r_3^2 = 16R^2 - a^2 - b^2 - c^2$.
- In an acute angled triangle ABC, $r + r_1 = r_2 + r_3$ and $\angle B > \frac{\pi}{3}$, then prove that $b + 3c < 3a < 3b + 3c$
- If the inradius in a right angled triangle with integer sides is r. Prove that
 - If $r = 4$, the greatest perimeter (in units) is 90
 - If $r = 5$, the greatest area (in sq. units) is 330
- If $\left(1 - \frac{r_1}{r_2}\right) \left(1 - \frac{r_1}{r_3}\right) = 2$, then prove that the triangle is right angled.



13. DEF is the triangle formed by joining the points of contact of the incircle with the sides of the triangle ABC; prove that
- (i) its sides are $2r \cos \frac{A}{2}$, $2r \cos \frac{B}{2}$ and $2r \cos \frac{C}{2}$,
- (ii) its angles are $\frac{\pi}{2} - \frac{A}{2}$, $\frac{\pi}{2} - \frac{B}{2}$ and $\frac{\pi}{2} - \frac{C}{2}$
and
- (iii) its area is $\frac{2\Delta^3}{(abc)s}$, i.e. $\frac{1}{2} \frac{r}{R} \Delta$.
14. Three circles, whose radii are a , b and c , touch one another externally and the tangents at their points of contact meet in a point, prove that the distance of this point from either of their points of contact is $\left(\frac{abc}{a+b+c} \right)^{\frac{1}{2}}$.
15. OA and OB are the equal sides of an isoscles triangle lying in the first quadrant making angles θ and ϕ respectively with x-axis. Show that the gradient of the bisector of acute angle AOB is $\operatorname{cosec} \beta - \cot \beta$ where $\beta = \phi + \theta$. (Where O is origin)
16. The hypotenuse $BC = a$ of a right-angled triangle ABC is divided into n equal segments where n is odd. The segment containing the midpoint of BC subtends angle α at A. Also h is the altitude of the triangle through A. Prove that $\tan \alpha = \frac{4nh}{a(n^2 - 1)}$.

Answers

5. $\angle ABD = 30^\circ$