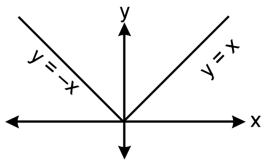
Fundamentals of Mathematics-II

He is unworthy of the name of man who is ignorant of the fact that the diagonal of square is incommensurable with its side ....**Plato**

Absolute value function / modulus function :

The symbol of modulus function is f (x) = $|x|$ and is defined as: y = $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$.



y = |x|

Properties of modulus : For any a, b $\in$ R

(i) $|a| \geq 0$ (ii) $|a| = |-a|$

(iii) $|a| \geq a$, $|a| \geq -a$ (iv) $|ab| = |a|\,|b|$

(v) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$ (vi) $|a + b| \leq |a| + |b|$; Equality holds when $ab \geq 0$

(vii) $|a - b| \geq ||a| - |b||$; Equality holds when $ab \geq 0$

Example # 1 : Solve the following linear equations
(i) $x|x| = 4$ (ii) $|x - 3| + 2|x + 1| = 4$

Solution : (i) $x|x| = 4$

If $x > 0$

$\therefore$ $x^2 = 4$ $\Rightarrow$ $x = \pm 2$

$\therefore$ $x = 2$ $(\because x \geq 0)$

If $x < 0$ $\Rightarrow$ $-x^2 = 4$ $\Rightarrow$ $x^2 = -4$ which is not possible

(ii) $|x - 3| + 2|x + 1| = 4$

case I : If $x \leq -1$

$\therefore$ $-(x - 3) - 2(x + 1) = 4$

$\Rightarrow$ $-x + 3 - 2x - 2 = 4$ $\Rightarrow$ $-3x + 1 = 4$

$\Rightarrow$ $-3x = 3$ $\Rightarrow$ $x = -1$

case II : If $-1 < x \leq 3$

$\therefore$ $-(x - 3) + 2(x + 1) = 4$

$\Rightarrow$ $-x + 3 + 2x + 2 = 4$ $\Rightarrow$ $x = -1$ which is not possible

case III : If $x > 3$

$x - 3 + 2(x + 1) = 4$

$3x - 1 = 4$ $\Rightarrow$ $x = 5/3$ which is not possible $\therefore$ $x = -1$ **Ans.**

Rational function :

A rational function is a function of the form, y = f (x) = $\frac{g(x)}{h(x)}$, where g (x) & h (x) are polynomial functions.

Irrational function :

An irrational function is a function y = f(x) in which the operations of addition, substraction, multiplication, division and raising to a fractional power are used.

For example y = $\frac{x^3 + x^{1/3}}{2x + \sqrt{x}}$ is an irrational function

(a) The equation $\sqrt{f(x)} = g(x)$, is equivalent to the following system

$f(x) = g^2(x)$ & $g(x) \geq 0$

(b) The inequation $\sqrt{f(x)} < g(x)$, is equivalent to the following system
$f(x) < g^2(x)$ & $f(x) \geq 0$ & $g(x) \geq 0$

(c) The inequation $\sqrt{f(x)} > g(x)$, is equivalent to the following system
$g(x) \leq 0$ & $f(x) \geq 0$ or $g(x) \geq 0$ & $f(x) > g^2(x)$

Example # 2 : **Solve :** $x + 2 > 2\sqrt{1-x^2}$

Solution : $4(1 - x^2) < (x + 2)^2$ and $x + 2 \geq 0$ & $1 - x^2 \geq 0$

$x \in \left(-\infty, \frac{-4}{5}\right) \cup (0, \infty)$ ...(1)

$x \in [-2, \infty)$ ...(2)

$x \in [-1, 1]$ ...(3)

$(1) \cap (2) \cap (3)$

$\left[-1, -\frac{4}{5}\right) \cup (0, 1]$

Self Practice Problem :

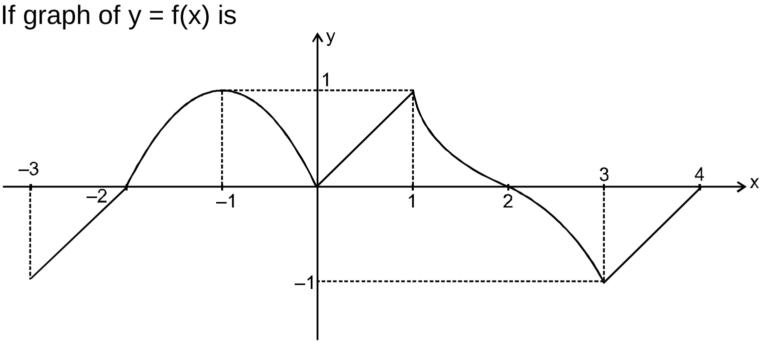
(1) $\sqrt{2x^2 + x - 6} < x$ (2) $\sqrt{5 - x} > x + 1$

(3) $x + 3 + \sqrt{x^2 + 4x - 5} > 0$ (4) $\sqrt{x} - \sqrt{4 - x} \geq 1$

Ans. (1) $\left[\frac{3}{2}, 2\right)$ (2) $(-\infty, 1)$ (3) $(-\infty, -1] \cup [5, \infty)$ (4) $\left[\frac{4+\sqrt{7}}{2}, 4\right]$

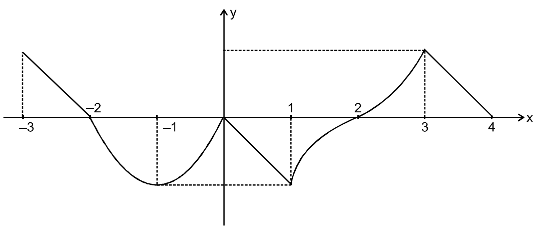
Graphs Related to modulus :

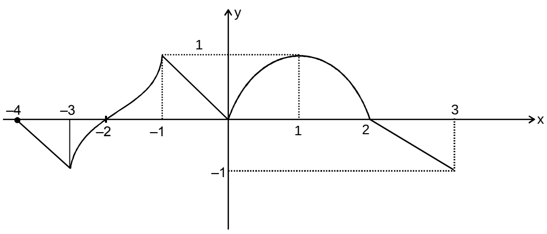
If graph of y = f(x) is

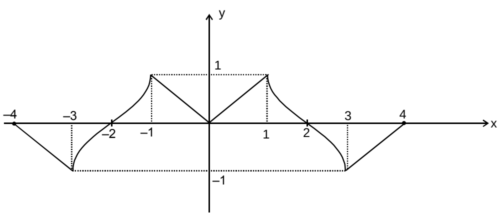


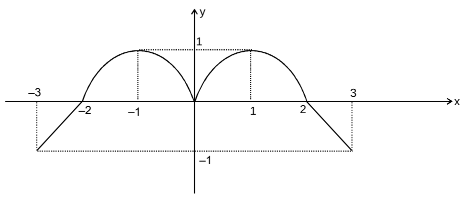
then draw graph of

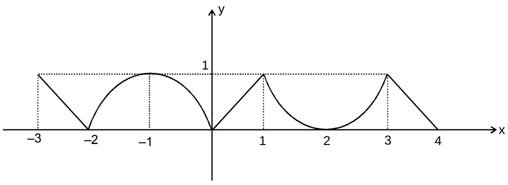
(a) $y = -f(x)$ (b) $y = f(-x)$ (c) $y = f(|x|)$ (d) $y = f(-|x|)$
(e) $y = |f(x)|$ (f) $|y| = f(x)$ (g) $|y| = -f(x)$

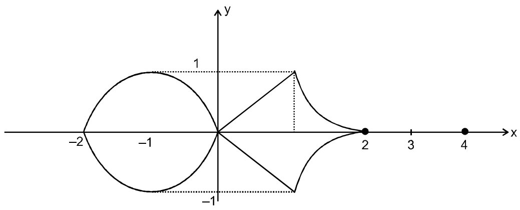
(a)


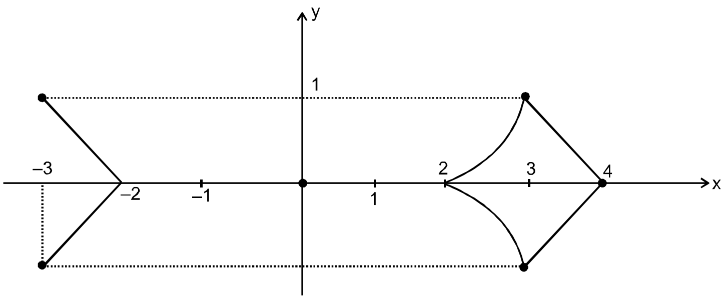
(b)


(c)


(d)


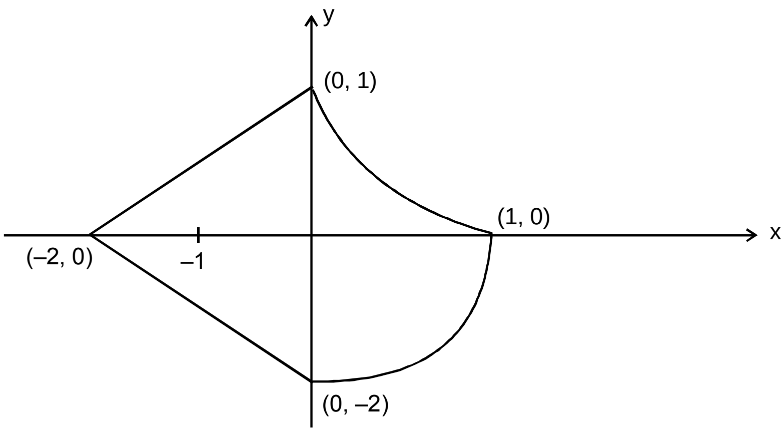
(e) 

(f) 

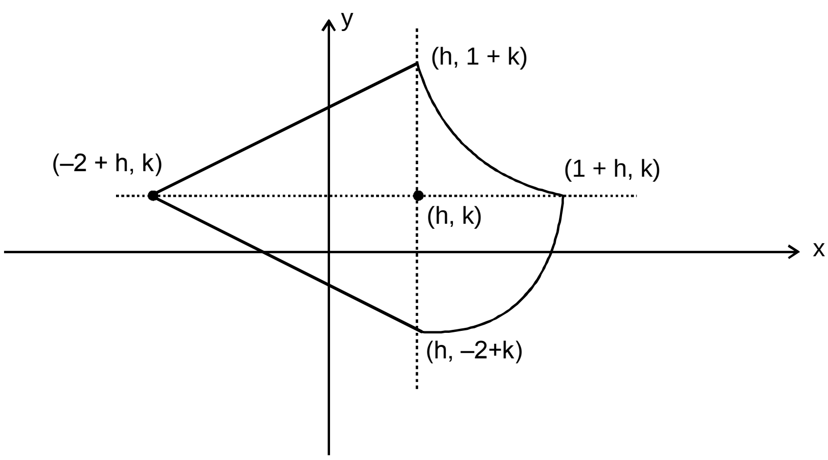
(g) 

Graphical Trasformation :

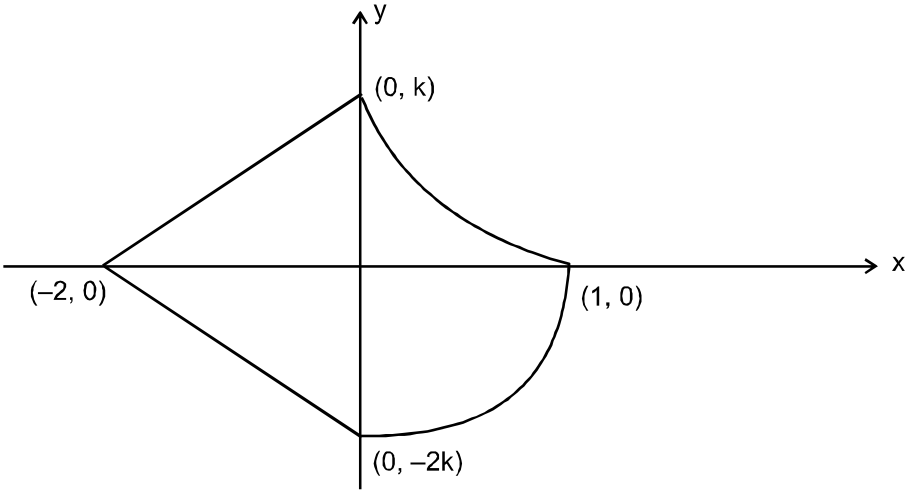
If graph of y = f(x) is



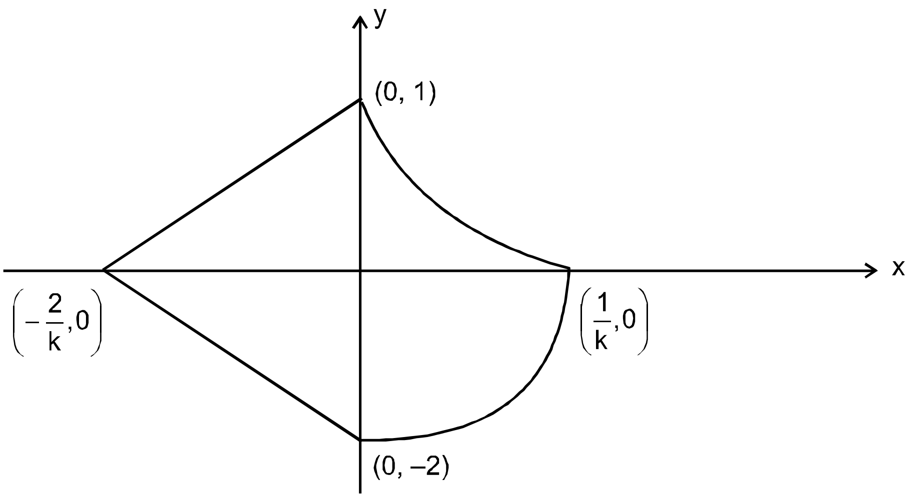
then graph of (a) $y - k = f(x - h)$ (b) $y = kf(x), (k > 0)$ (c) $y = f(kx), (k > 0)$

(a) 

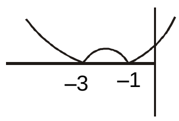
(b)



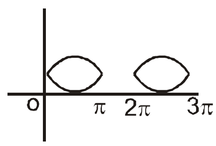
(c)



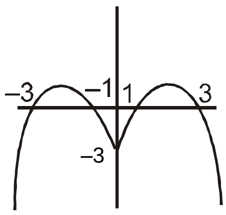
Example # 3 : $y = |x^2 + 4x + 3|$ **Solution :**



Example # 4 : $|y-1| = \sin x$ **Solution :**

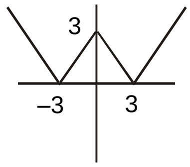


Example # 5 : $y = -x^2 + 4|x| - 3$ **Solution :**



Example # 6 : $y = ||x| - 3|$

Solution :



Example # 7 : $y = \sin\left(\frac{x}{3}\right)$

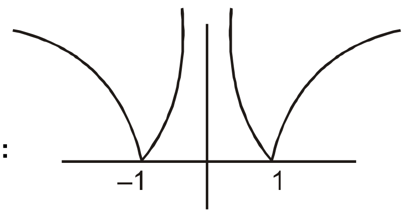
Solution : period is 6π

Example # 8 : $y = |\sin x - 3|$

Solution : Graphical Transformation

Example # 9 : $y = |-\ell n|-x||$

Solution :



Example # 10 : $|y| = x^2 - 3x + 2$

Solution :