Sequence & Series

"1729 is a very interesting number; it is the smallest number expressible as the sum of two cubes in two different ways." ...... S.Ramanujan

Sequence :

A sequence is a function whose domain is the set N of natural numbers. Since the domain for every sequence is the set N of natural numbers, therefore a sequence is represented by its range. If $f : N \to R$, then $f(n) = t_n$, $n \in N$ is called a sequence and is denoted by

$\{f(1), f(2), f(3), ...\} = \{t_1, t_2, t_3, ...\} = \{t_n\}$

Real sequence :

A sequence whose range is a subset of R is called a real sequence.

e.g. (i) 2, 5, 8, 11, ...

(ii) 4, 1, – 2, – 5, ...

Types of sequence :

On the basis of the number of terms there are two types of sequence.

(i) Finite sequences : A sequence is said to be finite if it has finite number of terms.

(ii) Infinite sequences : A sequence is said to be infinite if it has infinitely many terms.

Example # 1 : Write down the sequence whose n^{th} term is $\dfrac{(-2)^n}{(-1)^n + 2}$

Solution : Let $t_n = \dfrac{(-2)^n}{(-1)^n + 2}$

put n = 1, 2, 3, 4, ... we get

$t_1 = -2,\ t_2 = \dfrac{4}{3},\ t_3 = -8,\ t_4 = \dfrac{16}{3}$

so the sequence is $-2, , -8, \dfrac{16}{3}$, ...

Series :

By adding or subtracting the terms of a sequence, we get an expression which is called a series. If $a_1, a_2, a_3, ...a_n$ is a sequence, then the expression $a_1 + a_2 + a_3 + ... + a_n$ is a series.

e.g. (i) 1 + 2 + 3 + 4 + ... + n

(ii) 2 + 4 + 8 + 16 + ...

(iii) – 1 + 3 – 9 + 27 – ....

Progression :

The word progression refers to sequence or series – finite or infinite

Arithmetic progression (A.P.) :

A.P. is a sequence whose successive terms are obtained by adding a fixed number 'd' to the preceding terms. This fixed number 'd' is called the common difference. If a is the first term & d the common difference, then A.P. can be written as a, a + d, a + 2 d,....... ,a + (n – 1) d,........

e.g. – 4, – 1, 2, 5 ....

n^{th} term of an A.P. :

Let 'a' be the first term and 'd' be the common difference of an A.P., then

$t_n = a + (n - 1) d$, where $d = t_n - t_{n-1}$

Example # 2 : Find the number of terms in the sequence 4, 7, 10, 13, ...,82.

Solution : Let a be the first term and d be the common difference

a = 4, d = 3 so 82 = 4 + (n – 1)3

$\Rightarrow$ n = 27

The sum of first n terms of an A.P. :

If a is first term and d is common difference, then sum of the first n terms of AP is

$S_n = \frac{n}{2}\ [2a + (n - 1)\ d]$

$= \frac{n}{2}\ [a + \ell\ ] \equiv nt_{\left(\frac{n+1}{2}\right)}$, for n is odd. (Where ℓ is the last term and $t_{\left(\frac{n+1}{2}\right)}$ is the middle term.)

Note : For any sequence $\{t_n\}$, whose sum of first r terms is S_r, r^{th} term, $t_r = S_r - S_{r-1}$.

Example # 3 : If in an A.P., 3rd term is 18 and 7 term is 30, then find sum of its first 17 terms

Solution : Let a be the first term and d be the common difference

$a + 2d = 18$

$a + 6d = 30$

$d = 3$, $a = 12$

$s_{17} = \frac{17}{2}\ [2 \times 12 + 16 \times 3] = 612$

Example # 4 : Find the sum of all odd numbers between 1 and 1000 which are divisible by 3

Solution : Odd numbers between 1 and 1000 are

3, 5, 7, 9, 11, 13, ------ 993, 995, 997, 999.

Those numbers which are divisible by 3 are

3, 9, 15, 21, ------- 993, 999

They form an A.P. of which a = 3 , d = 6, $\ell = 999$ $\therefore$ n = 167

$S = \frac{n}{2}\ [a + \ell] = 83667$

Example # 5 : The ratio between the sum of n term of two A.P.'s is 3n + 8 : 7n + 15. Then find the ratio between their 12 th term

Solution : $\frac{S_n}{S_n'} = \frac{(n/2)[2a + (n-1)d]}{(n/2)[2a' + (n-1)d']} = \frac{3n+8}{7n+15}$ or $\frac{a + \{(n-1)/2\}d}{a' + (n-1)/2d'} = \frac{3n+8}{7n+15}$ ----- (i)

we have to find $\frac{T_{12}}{T_{12}'} = \frac{a + 11d}{a' + 11d'}$

choosing (n – 1)/2 = 11 or n = 23 in (1),

we get $\frac{T_{12}}{T_{12}'} = \frac{a + 11d}{a' + 11d'} = \frac{3(23) + 8}{(23) \times 7 + 15} = \frac{77}{176} = \frac{7}{16}$

Example # 6 : If sum of n terms of a sequence is given by $S_n = 3n^2 - 4n$, find its 50^{th} term.

Solution : Let t_n is n^{th} term of the sequence so $t_n = S_n - S_{n-1}$.

$= 3n^2 - 4n - 3(n - 1)^2 + 4(n - 1) = 6n - 7$

so $t_{50} = 293$.

Self practice problems :

(1) Which term of the sequence 2005, 2000, 1995, 1990, 1985, ... is the first negative term

(2) For an A.P. show that $t_m + t_{2n+m} = 2\ t_{m+n}$

(3) Find the maximum sum of the A.P. 40 + 38 + 36 + 34 + 32 + ...

(4) Find the sum of first 16 terms of an A.P. a_1, a_2, a_3..........
If it is known that $a_1 + a_4 + a_7 + a_{10} + a_{13} + a_{16} = 147$

Ans. (1) 403 (3) 420 (4) 392

Remarks :

(i) The first term and common difference can be zero, positive or negative (or any complex number.)

(ii) If a, b, c are in A.P. $\Rightarrow$ 2 b = a + c & if a, b, c, d are in A.P. $\Rightarrow$ a + d = b + c.

(iii) Three numbers in A.P. can be taken as a – d, a, a + d ; four numbers in A.P. can be taken as a – 3d, a – d, a + d, a + 3d; five numbers in A.P. are a – 2d, a – d, a, a + d, a + 2d ; six terms in A.P. are a – 5d, a – 3d, a – d, a + d, a + 3d, a + 5d etc.

(iv) The sum of the terms of an A.P. equidistant from the beginning & end is constant and equal to the sum of first & last terms.

(v) Any term of an A.P. (except the first) is equal to half the sum of terms which are equidistant from it. $a_n = 1/2\,(a_{n-k} + a_{n+k})$, k < n. For k = 1, $a_n = (1/2)\,(a_{n-1} + a_{n+1})$;
For k = 2, $a_n = (1/2)\,(a_{n-2} + a_{n+2})$ and so on.

(vi) If each term of an A.P. is increased, decreased, multiplied or divided by the same non-zero number, then the resulting sequence is also an AP.

(vii) The sum and difference of two AP's is an AP.

Example # 7 : The numbers $t\,(t^2 + 1)$, $-\frac{t^2}{2}$ and 6 are three consecutive terms of an A.P. If t be real, then find the the next two term of A.P.

Solution : $2b = a + c \quad \Rightarrow \quad -t^2 = t^3 + t + 6$
or $\quad t^3 + t^2 + t + 6 = 0$
or $\quad (t + 2)\,(t^2 - t + 3) = 0 \qquad \therefore \quad t^2 - t + 3 \neq 0 \quad \Rightarrow \quad t = -2$
the given numbers are – 10, – 2, 6
which are in an A.P. with d = 8. The next two numbers are 14, 22

Example # 8 : If a_1, a_2, a_3, a_4, a_5 are in A.P. with common difference $\neq 0$, then find the value of $\sum_{i=1}^{5} a_i$, when $a_3 = 2$.

Solution : As a_1, a_2, a_3, a_4, a_5 are in A.P., we have $a_1 + a_5 = a_2 + a_4 = 2a_3$.
Hence $\sum_{i=1}^{5} a_i = 10$.

Example # 9 : If a(b + c), b(c + a), c(a + b) are in A.P., prove that $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ are also in A.P.

Solution : $\because$ a(b + c), b(c + a), c(a + b) are in A.P. $\Rightarrow$ subtract ab + bc + ca from each
– bc, – ca, – ab are in A.P.
divide by –abc
$\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ are in A.P.

Example # 10 : If $\frac{a+b}{1-ab}$, b, $\frac{b+c}{1-bc}$ are in A.P. then prove that $\frac{1}{a}$, b $\frac{1}{c}$ are in A.P.

Solution : $\therefore \quad \frac{a+b}{1-ab}$, b, $\frac{b+c}{1-bc}$ are in A.P.

$$b - \frac{a+b}{1-ab} = \frac{b+c}{1-bc} - b$$

$$\frac{-a\left(b^2+1\right)}{1-ab} = \frac{c\left(1+b^2\right)}{1-bc}$$

$\Rightarrow \quad -a + abc = c - abc$
a + c = 2abc
divide by ac

$\frac{1}{c} + \frac{1}{a} = 2b \quad \Rightarrow \quad \frac{1}{a}$, b, $\frac{1}{c}$ are in A.P.

Arithmetic mean (mean or average) (A.M.) :

If three terms are in A.P. then the middle term is called the A.M. between the other two, so if a, b, c are in A.P., b is A.M. of a & c.

A.M. for any n numbers $a_1, a_2,..., a_n$ is; $A = \frac{a_1 + a_2 + a_3 + ... + a_n}{n}$.

n-Arithmetic means between two numbers :

If a, b are any two given numbers & a, $A_1, A_2,...., A_n$, b are in A.P., then $A_1, A_2,... A_n$ are the n A.M.'s between a & b.

$A_1 = a + \frac{b-a}{n+1}$, $A_2 = a + \frac{2(b-a)}{n+1}$,......, $A_n = a + \frac{n(b-a)}{n+1}$

Note : Sum of n A.M.'s inserted between a & b is equal to n times the single A.M. between a & b

i.e. $\sum_{r=1}^{n} A_r = nA$, where A is the single A.M. between a & b   i.e. $A = \frac{a+b}{2}$

Example # 11 : If a, b, c,d,e, f are A. M's between 2 and 12, then find a + b + c + d + e + f.

Solution : Sum of A.M.^{s} = 6 single A.M. $= \frac{6(2+12)}{2} = 42$

Example # 12 : Insert 10 A.M. between 3 and 80.

Solution : Here 3 is the first term and 80 is the 12^{th} term of A.P. so 80 = 3 + (11)d

$\Rightarrow$ d = 7

so the series is 3, 10, 17, 24, ..., 73, 80

$\therefore$ required means are 10, 17, 24, ..., 73.

Self practice problems :

(5) There are n A.M.'s between 3 and 29 such that 6th mean : (n – 1)th mean : : 3 : 5 then find the value of n.

(6) For what value of n, $\frac{a^{n+3} + b^{n+3}}{a^{n+2} + b^{n+2}}$, $a \neq b$ is the A.M. of a and b.

Ans. (5) n = 12 (6) n = –2

Geometric progression (G.P.) :

G.P. is a sequence of numbers whose first term is non zero & each of the succeeding terms is equal to the preceeding terms multiplied by a constant. Thus in a G.P. the ratio of successive terms is constant. This constant factor is called the **common ratio** of the series & is obtained by dividing any term by that which immediately preceeds it. Therefore a, ar, ar^2, ar^3, ar^4,...... is a G.P. with 'a' as the first term & 'r' as common ratio.

e.g. (i) 2, 4, 8, 16, ... (ii) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}$, ...

Results : (i) n^{th} term of GP = $a\, r^{n-1}$

(ii) Sum of the first n terms of GP

$$S_n = \begin{cases} \frac{a(r^n - 1)}{r-1} & , \quad r \neq 1 \\ na & , \quad r = 1 \end{cases}$$

(iii) Sum of an infinite terms of GP when $|r| < 1$. When $n \to \infty$, $r^n \to 0$ if $|r| < 1$ therefore,

$S_\infty = \frac{a}{1-r} \; (|r| < 1)$

Example # 13 : The n^{th} term of the series 3, $\sqrt{3}$, 1 , ------ is $\frac{1}{243}$, then find n

Solution : $3.\left(\frac{1}{\sqrt{3}}\right)^{n-1} = \frac{1}{243} \Rightarrow n = 13$

Example # 14 : The first term of an infinite G.P. is 1 and any term is equal to the sum of all the succeeding terms. Find the series.

Solution : Let the G.P. be 1, r, r^2, r^3, ...

given condition $\Rightarrow$ $r = \frac{r^2}{1-r}$ $\Rightarrow$ $r = \frac{1}{2}$,

Hence series is 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, ...∞

Example # 15 : In a G.P.., $T_2 + T_5 = 216$ and $T_4 : T_6 = 1 : 4$ and all terms are integers, then find its first term :

Solution : $ar(1 + r^3) = 216$ and $\frac{ar^3}{ar^5} = \frac{1}{4}$

$\Rightarrow$ $r^2 = 4 \Rightarrow r = \pm 2$

when $r = 2$ then $2a(9) = 216 \Rightarrow a = 12$

when $r = -2$, then $-2a(1-8) = 216$

$\therefore$ $a = \frac{216}{14} = \frac{108}{7}$, which is not an integer.

Self practice problems :

(7) Find the G.P. if the common ratio of G.P. is 3, n^{th} term is 486 and sum of first n terms is 728.

(8) If x, 2y, 3z are in A.P. where the distinct numbers x, y, z are in G.P. Then find the common ratio of G.P.

(9) A G.P. consist of 2n terms. If the sum of the terms occupying the odd places is S_1 and that of the terms occupying the even places is S_2 , then find the common ratio of the progression.

(10) If continued prodcut of three number in G.P. is 216 and sum of there product in pairs is 156. Find the numbers.

Ans. (7) 2, 6, 18, 54, 162, 486 (8) $\frac{1}{3}$ (9) $\frac{S_2}{S_1}$.

(10) 2, 6, 18

Remarks :

(i) If a, b, c are in G.P. $\Rightarrow b^2 = ac$, in general if a_1, a_2, a_3, a_4,......... a_{n-1} , a_n are in G.P., then $a_1a_n = a_2a_{n-1} = a_3\, a_{n-2}$ = ...

(ii) Any three consecutive terms of a G.P. can be taken as $\frac{a}{r}$, a , ar.

(iii) Any four consecutive terms of a G.P. can be taken as, $\frac{a}{r^3}$, $\frac{a}{r}$ ar, ar^3.

(iv) If each term of a G.P. be multiplied or divided or raised to power by the same non–zero quantity, the resulting sequence is also a G.P..

(v) If a_1, a_2, a_3,........ and b_1, b_2, b_3,......... are two G.P's with common ratio r_1 and r_2 respectively, then the sequence a_1b_1, a_2b_2, a_3b_3, ... is also a G.P. with common ratio $r_1\, r_2$.

(vi) If a_1, a_2, a_3,..........are in G.P. where each $a_i > 0$, then log a_1, loga_2, loga_3,..........are in A.P. and its converse is also true.

Example # 16 : Three numbers form an increasing G.P. If the middle number is doubled, then the new numbers are in A.P. The common ratio of G.P. is :

Solution : Three number in G.P. are $\frac{a}{r}$, a, ar

then $\frac{a}{r}$, 2a ar are in A.P. as given.

$\therefore$ $2(2a) = a\left(r + \frac{1}{r}\right)$

or $r^2 - 4r + 1 = 0$

or $r = 2 \pm \sqrt{3}$

or $r = 2 + \sqrt{3}$ as r > 1 for an increasing G.P.

Example # 17 : The sum of an infinite geometric progression is 2 and the sum of the geometric progression made from the cubes of this infinite series is 24. Then find its first term and common ratio :

Solution : Let a be the first term and r be the common ratio of G.P.

$\frac{a}{1-r} = 2, \frac{a^3}{1-r^3} = 24$, $-1 < r < 1$

Solving we get a = 3, $r = -\frac{1}{2}$

Example # 18 : Express $0.4\dot{2}\dot{3}$ in the form of $\frac{p}{q}$, (where $p, q \in I, q \neq 0$)

Solution : $S = \frac{4}{10} + \frac{23}{10^3} + \frac{23}{10^5} + ... \infty = \frac{4}{10} + \frac{a}{1-r} = \frac{4}{10} + \frac{23}{990} = \frac{419}{990}$

Example # 19 : Evaluate 9 + 99 + 999 + ... upto n terms.

Solution : Let S = 9 + 99 + 999 + ...upto n terms.

= [9 + 99 + 999 + ...]

$= [(10 - 1) + (10^2 - 1) + (10^3 - 1) + ... + \text{upto n terms}]$

$= [10 + 10^2 + 10^3 + ...+ 10^n - n] = \left(\frac{10(10^n - 1)}{9} - n\right)$

Geometric means (mean proportional) (G.M.):

If a, b, c are in G.P., b is called as the G.M. of a & c.

If a and c are both positive, then $b = \sqrt{ac}$ and if a and c are both negative, then $b = -\sqrt{ac}$.

$b^2 = ac$, therefore $b = \sqrt{ac}$; a > 0, c > 0.

n-Geometric means between a, b :

If a, b are two given numbers & a, $G_1, G_2, ..., G_n$, b are in G.P.. Then

$G_1, G_2, G_3, ..., G_n$ are n G.M.s between a & b.

$G_1 = a(b/a)^{1/n+1}, G_2 = a(b/a)^{2/n+1}, ..., G_n = a(b/a)^{n/n+1}$

Note : The product of n G.M.s between a & b is equal to the nth power of the single G.M. between a & b

i.e. $\prod_{r=1}^{n} G_r = \left(\sqrt{ab}\right)^n = G^n$, where G is the single G.M. between a & b.

Example # 20 : Between 4 and 2916 are inserted odd number (2n + 1) G.M's. Then the (n + 1)th G.M. is

Solution : $4, G_1.G_2, ... G_{n+1}, ... G_{2n}, G_{2n+1}, 2916$

G_{n+1} will be the middle mean of (2n +1) odd means and it will be equidistant from Ist and last term

$\therefore$ $4, G_{n+1}$, 2916 will also be in G.P.

$\therefore$ $G_{n+1}^2 = 4 \times 2916 = 4 \times 9 \times 324 = 4 \times 9 \times 4 \times 81$

$G_{n+1} = 2 \times 3 \times 2 \times 9 = 108.$

Self practice problems :

(11) Find the value of n so that $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ may be the G.M. between a and b.

(12) If a = $\underbrace{111\ ....1}_{55}$, b = 1 + 10 + 10^2 + 10^3 + 10^4 and c = 1 + 10^5 + 10^{10} + ... + 10^{50}, then prove that

(i) 'a' is a composite number (ii) a = bc.

Ans. (11) $n = -\frac{1}{2}$

Harmonic progression (H.P.)

A sequence is said to be in H.P if the reciprocals of its terms are in A.P.. If the sequence a_1, a_2, a_3,...., a_n is in H.P. then $1/a_1$, $1/a_2$,...., $1/a_n$ is in A.P.

Note : (i) Here we do not have the formula for the sum of the n terms of an H.P.. For H.P. whose first term is a and second term is b, the n^{th} term is $t_n = \frac{ab}{b+(n-1)(a-b)}$.

(ii) If a, b, c are in H.P. $\Rightarrow b = \frac{2ac}{a+c}$ or $\frac{a}{c} = \frac{a-b}{b-c}$.

(iii) If a, b, c are in A.P. $\Rightarrow \frac{a-b}{b-c} = \frac{a}{a}$

(iv) If a, b, c are in G.P. $\Rightarrow \frac{a-b}{b-c} = \frac{a}{b}$

Harmonic mean (H.M.):

If a, b, c are in H.P., b is called as the H.M. between a & c, then $b = \frac{2ac}{a+c}$

If a_1, a_2 , ... a_n are 'n' non-zero numbers then H.M. 'H' of these numbers is given by

$$\frac{1}{H} = \frac{1}{n}\left[\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}\right]$$

Example # 21 : The 7th term of a H.P. is $\frac{1}{10}$ and 12th term is $\frac{1}{25}$, find the 20th term of H.P.

Solution : Let a be the first term and d be the common difference of corresponding A.P.

a + 6d = 10

a + 11d = 25

5d = 15

d = 3, a = – 8

T_{20} = a + 19d

= – 8 + 19 × 3 = 49

20 term of H.P. = $\frac{1}{49}$

Example # 22 : Insert 4 H.M between $\frac{3}{4}$ and $\frac{3}{19}$.

Solution : Let 'd' be the common difference of corresponding A.P..

so $d = \frac{\frac{19}{3} - \frac{4}{3}}{5} = 1.$

$\therefore \quad \frac{1}{H_1} = \frac{4}{3} + 1 = \frac{7}{3}$ or $H_1 = \frac{3}{7}$

$\frac{1}{H_2} = \frac{4}{3} + 2 = \frac{10}{3}$ or $H_2 = \frac{3}{10}$

$\frac{1}{H_3} = \frac{4}{3} + 3 = \frac{13}{3}$ or $H_3 = \frac{3}{13}$

$\frac{1}{H_4} = \frac{4}{3} + 4 = \frac{16}{3}$ or $H_4 = \frac{3}{16}$.

Example # 23 : Find the largest positive term of the H.P., whose first two term are $\frac{2}{5}$ and $\frac{12}{23}$.

Solution : The corresponding A.P. is $\frac{5}{2}$, $\frac{23}{12}$ ....or $\frac{30}{12}$, $\frac{23}{12}$, $\frac{16}{12}$, $\frac{9}{12}$, $\frac{2}{12}$, $\frac{-5}{12}$ ....

The H.P. is $\frac{12}{30}$, $\frac{12}{23}$, $\frac{12}{16}$, $\frac{12}{9}$, $\frac{12}{2}$, $-\frac{12}{5}$, ...

Largest positive term = $\frac{12}{2} = 6$

Self practice problems :

(13) If a, b, c, d, e are five numbers such that a, b, c are in A.P., b, c, d are in G.P. and c, d, e are in H.P. prove that a, c, e are in G.P.

(14) If the ratio of H.M. between two positive numbers 'a' and 'b' (a > b) is to their G.M. as 12 to 13, prove that a : b is 9 : 4.

(15) a, b, c are in H.P. then prove that $\frac{b+a}{b-a} + \frac{b+c}{b-c} = 2$

(16) If a, b, c, d are in H.P., then show that ab + bc + cd = 3ad

Arithmetico-geometric series :

A series, each term of which is formed by multiplying the corresponding terms of an A.P. & G.P. is called the Arithmetico-Geometric Series. e.g. $1 + 3x + 5x^2 + 7x^3 + ...$

Here 1, 3, 5,.... are in A.P. & 1, x, x^2, x^3.... are in G.P..

Sum of n terms of an arithmetico-geometric series:

Let $S_n = a + (a + d) r + (a + 2 d) r^2 + ... + [a + (n-1)d] r^{n-1}$, then

$$S_n = \frac{a}{1-r} + \frac{dr\left(1-r^{n-1}\right)}{(1-r)^2} - \frac{[a+(n-1)d]r^n}{1-r}, \ r \neq 1.$$

Sum to infinity: If $|r| < 1$ & $n \to \infty$, then $\lim\limits_{n\to\infty} r^n = 0$ and $\lim\limits_{n\to\infty} n.r^n = 0$

$$\therefore \quad \mathbf{S}_\infty = \frac{a}{1-r} + \frac{dr}{(1-r)^2}.$$

Example # 24 : The sum to n terms of the series $1 + 5\left(\frac{4n+1}{4n-3}\right) + 9\left(\frac{4n+1}{4n-3}\right)^2 + 13\left(\frac{4n+1}{4n-3}\right)^3 + ...$ is .

Solution : Let $x = \frac{4n+1}{4n-3}$, then

$1 - x = \frac{-4}{4n-3}$, $\frac{1}{1-x} = -\frac{(4n-3)}{4}$

$\frac{x}{1-x} = -\frac{(4n+1)}{4}$

$S = 1 + 5x + 9x^2 + ... + (4n-3)x^{n-1}$

$Sx = x + 5x^2 + ... (4n-3)x^n$

$S - Sx = 1 + 4x + 4x^2 + ... + 4x^{n-1} - (4n-3)x^n$.

$S(1-x) = 1 + \frac{4x}{1-x}\ [1 - x^{n-1}] - (4n-3)x^n$

$$S = \frac{1}{1-x}\left[1 + \frac{4x}{1-x} - \frac{4x^n}{1-x} - (4n-3)x^n\right] = -\frac{(4n-3)}{4}[1-(4n+1)+(4n-3)x^n-(4n-3)x^n] = n\,(4n-3).$$

Example # 25 : Find sum to infinite terms of the series $1 + 2x + 3x^2 + 4x^3 + ...,\ -1 < x < 1$

Solution : let $S = 1 + 2x + 3x^2 + 4x^3 + ...$ ....(i)

$xS = x + 2x^2 + 3x^3 + ...$ ....(ii)

(i) - (ii) $\Rightarrow$ $(1 - x) S = 1 + x + x^2 + x^3 + ...$

or $S = \dfrac{1}{(1-x)^2}$

Example # 26 : Evaluate : $1^2 + 2^2x + 3^2x^2 + 4^2x^3 ....$ upto infinite terms for $| x | < 1$.

Solution : Let $s = 1^2 + 2^2x + 3^2x^2 + 4^2x^3 .... \infty$ ...(i)

$xs = 1^2x + 2^2x^2 + 3^2x^3 .... \infty$ ...(ii)

(i) – (ii)

$(1 - x) s = 1 + 3x + 5x^2 + 7x^3 + ...$

$(1 - x) s = \dfrac{1}{1-x} + \dfrac{2x}{(1-x)^2}$

$s = \dfrac{1}{(1-x)^2} + \dfrac{2x}{(1-x)^3}$

$s = \dfrac{1-x+2x}{(1-x)^3}$

$s = \dfrac{1+x}{(1-x)^3}$

Self practice problems :

(17) If $4 + \dfrac{4+d}{5} + \dfrac{4+2d}{5^2}$ .... = 1, then find d.

(18) Evaluate : $1 + 3x + 6x^2 + 10x^3 + ...$ upto infinite term, where $| x | < 1$.

(19) Sum to n terms of the series : $1 + 2\left(1+\dfrac{1}{n}\right) + 3\left(1+\dfrac{1}{n}\right)^2 + ...$

Ans. (17) $-\dfrac{64}{5}$

(18) $\dfrac{1}{(1-x)^3}$

(19) n^2

Relation between means :

(i) If A, G, H are respectively A.M., G.M., H.M. between a & b both being positive, then $G^2 = AH$ (i.e. A, G, H are in G.P.) and $A \ge G \ge H$.

Example # 27 : The A.M. of two numbers exceeds the G.M. by 2 and the G.M. exceeds the H.M. by $\dfrac{8}{5}$; find the numbers.

Solution : Let the numbers be a and b, now using the relation

$G^2 = AH = (G + 2)\left(G - \dfrac{8}{5}\right) \Rightarrow G = 8$; $A = 10$

i.e. $ab = 64$

also $a + b = 20$

Hence the two numbers are 4 and 16.

A.M. ≥ G.M. ≥ H.M.

Let $a_1, a_2, a_3, ...a_n$ be n positive real numbers, then we define their

A.M. = $\frac{a_1 + a_2 + a_3 + ... + a_n}{n}$, their

G.M. = $(a_1 a_2 a_3 ....a_n)^{1/n}$ and their

H.M. = $\frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + ... + \frac{1}{a_n}}$.

It can be shown that A.M. ≥ G.M. ≥ H.M. and equality holds at either places iff $a_1 = a_2 = a_3 = ... = a_n$

Example # 28 : If a, b, c > 0, prove that $\frac{ab}{c^2} + \frac{bc}{a^2} + \frac{ca}{b^2} \geq 3$

Solution : Using the relation A.M. ≥ G.M. we have

$$\frac{\frac{ab}{c^2} + \frac{bc}{a^2} + \frac{ca}{b^2}}{3} \geq \left(\frac{ab}{c^2}.\frac{bc}{a^2}.\frac{ca}{b^2}\right)^{\frac{1}{3}} \Rightarrow \frac{ab}{c^2} + \frac{bc}{a^2} + \frac{ca}{b^2} \geq 3$$

Example # 29 : If $a_i > 0 \ \forall \ i = 1, 2, 3, ...$ prove that $(a_1 + a_2 + a_3 .... + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} ..... + \frac{1}{a_n}\right) \geq n^2$

Solution : Using the relation A.M. ≥ H.M.

$$\frac{a_1 + a_2 + a_3 .... + a_n}{n} \geq \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} ..... + \frac{1}{a_n}}$$

$$\Rightarrow \quad (a_1 + a_2 + a_3 .... + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} ..... + \frac{1}{a_n}\right) \geq n^2$$

Example # 30 : If x, y, z are positive then prove that $(x + y)(y + z)(z + x)\left(\frac{1}{x} + \frac{1}{y}\right)\left(\frac{1}{y} + \frac{1}{z}\right)\left(\frac{1}{z} + \frac{1}{x}\right) \geq 64$

Solution : Using the relation A.M. ≥ H.M.

$$\frac{x + y}{2} \geq \frac{2}{\frac{1}{x} + \frac{1}{y}} \Rightarrow \quad (x + y)\left(\frac{1}{x} + \frac{1}{y}\right) \geq 4 \quad .....(i)$$

similarly $(y + z)\left(\frac{1}{y} + \frac{1}{z}\right) \geq 4$ .....(ii)

$(z + x) \geq 4\left(\frac{1}{z} + \frac{1}{x}\right)$ .....(iii)

by (i), (ii) & (iii) $(x + y)(y + z)(z + x)\left(\frac{1}{x} + \frac{1}{y}\right)\left(\frac{1}{y} + \frac{1}{z}\right)\left(\frac{1}{z} + \frac{1}{x}\right) \geq 64$

Example # 31 : If n > 0, prove that $2^n > 1 + n\sqrt{2^{n-1}}$

Solution : Using the relation A.M. ≥ G.M. on the numbers $1, 2, 2^2, 2^3, ..., 2^{n-1}$, we have

$$\frac{1 + 2 + 2^2 + ... + 2^{n-1}}{n} > (1.2.\ 2^2.\ 2^3.\ ....\ .2^{n-1})^{1/n}$$

Equality does not hold as all the numbers are not equal.

$$\Rightarrow \quad \frac{2^n - 1}{2 - 1} > n\left(2^{\frac{(n-1)n}{2}}\right)^{\frac{1}{n}} \quad \Rightarrow \quad 2^n - 1 > n\ 2^{\frac{(n-1)}{2}}$$

$$\Rightarrow \quad 2^n > 1 + n\,2^{\frac{(n-1)}{2}}.$$

Example # 32 : If x, y, z are positive and x + y + z = 7 then find greatest value of $x^2 y^3 z^2$.

Solution : Using the relation A.M. ≥ G.M.

$$\frac{\frac{x}{2}+\frac{x}{2}+\frac{y}{3}+\frac{y}{3}+\frac{y}{3}+\frac{z}{2}+\frac{z}{2}}{7} \geq \left(\frac{x^2}{4}.\frac{y^3}{27}.\frac{z^2}{4}\right)^{\frac{1}{7}}$$

$$\Rightarrow \quad 1 \geq \left(\frac{x^2}{4}.\frac{y^3}{27}.\frac{z^2}{4}\right)^{\frac{1}{7}} \quad \Rightarrow \quad 432 \geq x^2 y^3 z^2$$

Self practice problems :

(20) If a, b, c are real and distinct, then show that $a^2 (1 + b^2) + b^2 (1 + c^2) + c^2 (1 + a^2) > 6abc$

(21) Prove that $2.4.6.8.......2n < (n + 1)^n$. $(n \in N)$

(22) If a, b, c, d are positive real numbers prove that $\frac{bcd}{a^2}+\frac{cda}{b^2}+\frac{dab}{c^2}+\frac{abc}{d^2} > a + b + c + d$

(23) If $x^6 - 12x^5 + ax^4 + bx^3 + cx^2 + dx + 64 = 0$ has positive roots then find a, b, c, d,

(24) If a, b > 0, prove that $[(1 + a)(1 + b)]^3 > 3^3 a^2 b^2$

Ans. (23) a = 60, b = –160, c = 240, d = –192

Results :

(i) $\sum_{r=1}^{n} (a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$. (ii) $\sum_{r=1}^{n} k\, a_r = \sum_{r=1}^{n} k\, a_r$.

(iii) $\sum_{r=1}^{n} k = k + k + k + ...n$ times $= nk$; where k is a constant.

(iv) $\sum_{r=1}^{n} r = 1 + 2 + 3 + ...+ n = \frac{n\,(n+1)}{2}$

(v) $\sum_{r=1}^{n} r^2 = 1^2 + 2^2 + 3^2 + ...+ n^2 = \frac{n\,(n+1)\,(2n+1)}{6}$

(vi) $\sum_{r=1}^{n} r^3 = 1^3 + 2^3 + 3^3 + ...+ n^3 = \frac{n^2\,(n+1)^2}{4}$

Example # 33 : Find the sum of the series to n terms whose n^{th} term is 3n + 2.

Solution : $S_n = \Sigma T_n = \Sigma(3n + 2) = 3\Sigma n + \Sigma 2 \quad = \frac{3(n+1)\;n}{2} + 2n = \frac{n}{2}\,(3n + 7)$

Example # 34 : $T_k = k^3 + 3^k$, then find $\sum_{k=1}^{n} T_k$.

Solution : $\sum_{k=1}^{n} T_k = \sum_{k=1}^{n} k^3 + \sum_{k=1}^{n} 3^k = \left(\frac{n(n+1)}{2}\right)^2 + \frac{3(3^n-1)}{3-1} = \left(\frac{n(n+1)}{2}\right)^2 + \frac{3}{2}\,(3^n - 1)$

Method of difference for finding n^{th} term :

Let u_1, u_2, u_3 .... be a sequence, such that $u_2 - u_1, u_3 - u_2$, ... is either an A.P. or a G.P. then nth term u_n of this sequence is obtained as follows

$S = u_1 + u_2 + u_3 + \ldots\ldots\ldots + u_n$ ....(i)

$S = \quad u_1 + u_2 + \ldots\ldots\ldots + u_{n-1} + u_n$ ....(ii)

(i) – (ii) $\Rightarrow$ $u_n = u_1 + (u_2 - u_1) + (u_3 - u_2) + \ldots\ldots\ldots + (u_n - u_{n-1})$

Where the series $(u_2 - u_1) + (u_3 - u_2) + \ldots\ldots\ldots + (u_n - u_{n-1})$ is

either in A.P. or in G.P. then we can find u_n. So sum of series $S = \sum_{r=1}^{n} u_r$

Note : The above method can be generalised as follows :

Let u_1, u_2, u_3, ... be a given sequence.

The first differences are $\Delta_1u_1, \Delta_1u_2, \Delta_1u_3$, ... where $\Delta_1u_1 = u_2 - u_1$, $\Delta_1u_2 = u_3 - u_2$ etc.

The second differences are $\Delta_2u_1, \Delta_2u_2, \Delta_2u_3$, ..., where $\Delta_2u_1 = \Delta_1u_2 - \Delta_1u_1$, $\Delta_2u_2 = \Delta_1u_3 - \Delta_1u_2$ etc.

This process is continued untill the k^{th} differences Δ_ku_1 , Δ_ku_2 , ... are obtained, where the k^{th} differences are all equal or they form a GP with common ratio different from 1.

Case - 1 : The k^{th} differences are all equal.

In this case the n^{th} term, u_n is given by

$u_n = a_0n^k + a_1n^{k-1} + \ldots\ldots + a_k$, where $a_0, a_1, \ldots., a_k$ are calculated by using first 'k + 1' terms of the sequence.

Case - 2 : The k^{th} differences are in GP with common ratio r $(r \neq 1)$

The n^{th} term is given by $u_n = \lambda\, r^{n-1} + a_0\, n^{k-1} + a_1\, n^{k-2} + \ldots\ldots + a_{k-1}$

Example # 35 : Find the n^{th} term of the series 1, 3, 8, 16, 27, 41, ...

Solution : $s = 1 + 3 + 8 + 16 + 27 + 41 + \ldots\ldots T_n$ .....(i)

$s = \quad 1 + 3 + 8 + 16 + 27 \ldots\ldots T_{n-1} + T_n$ .....(ii)

(i) – (ii)

$T_n = 1 + 2 + 5 + 8 + 11 + \ldots\ldots (T_n - T_{n-1})$

$T_n = 1 + \left(\frac{n-1}{2}\right) [2 \times 2 + (n-2)3] = \frac{1}{2} [3n^2 - 5n + 4]$

Example # 36 : Find the sum to n terms of the series 5, 7, 13, 31, 85 + ...

Solution : Successive difference of terms are in G.P. with common ratio 3.

$T_n = a(3)^{n-1} + b$

$a + b = 5$

$3a + b = 7 \quad \Rightarrow \quad a = 1, b = 4$

$T_n = 3^{n-1} + 4$

$S_n = \Sigma T_n = \Sigma(3^{n-1} + 4) = (1 + 3 + 3^2 + \ldots\ldots + 3^{n-1}) + 4n$

$\frac{1}{2} [3^n + 8n - 1]$

Method of difference for finding s_n :

If possible express r^{th} term as difference of two terms as $t_r = \pm (f(r) - f(r \pm 1))$. This can be explained with the help of examples given below.

$t_1 = f(1) - f(0)$,

$t_2 = f(2) - f(1)$,

$\vdots \quad \vdots \quad \vdots$

$t_n = f(n) - f(n-1)$

$\Rightarrow \quad S_n = f(n) - f(0)$

Example # 37 : Find the sum of n-terms of the series 2.5 + 5.8 + 8.11 + ...

Solution : $T_r = (3r - 1)(3r + 2) = 9r^2 + 3r - 2$

$$S_n = \sum_{r=1}^{n} T_r = 9\sum_{r=1}^{n} T_r + 3\sum_{r=1}^{n} r - \sum_{r=1}^{n} 2$$

$$= 9\left(\frac{n(n+1)(2n+1)}{6}\right) + 3\left(\frac{n(n+1)}{2}\right) - 2n$$

$$= 3n(n+1)^2 - 2n$$

Example # 38 : Sum to n terms of the series $\frac{1}{(1+x)(1+3x)} + \frac{1}{(1+3x)(1+5x)} + \frac{1}{(1+5x)(1+7x)} + ...$

Solution : Let T_r be the general term of the series

$$T_r = \frac{1}{[1+(2r-1)x][1+(2r+1)x]}$$

So $\quad T_r = \frac{1}{2x}\left[\frac{(1+(2r+1)x)-(1+(2r-1)x)}{(1+(2r-1)x)(1+(2r+1)x)}\right] = \left[\frac{1}{(1+(2r-1)x)} - \frac{1}{(1+(2r+1)x)}\right]$

$\therefore \quad S_n = \sum T_r = T_1 + T_2 + T_3 + ... + T_n$

$$= \frac{1}{2x}\left[\frac{1}{1+x} - \frac{1}{(1+(2n+1)x)}\right] = \frac{n}{(1+x)[1+(2n+1)x]}$$

Example # 39 : Sum to n terms of the series $\frac{1}{1.4.7} + \frac{1}{4.7.10} + \frac{1}{7.10.13} + ...$

Solution : $T_n = \frac{1}{(3n-2)(3n+1)(3n+4)} = \frac{1}{6}\left[\frac{1}{(3n-2)(3n+1)} - \frac{1}{(3n+1)(3n+4)}\right]$

$$= \frac{1}{6}\left[\left(\frac{1}{1.4} - \frac{1}{4.7}\right) + \left(\frac{1}{4.7} - \frac{1}{7.10}\right) + ... + \frac{1}{(3n-2)(3n+1)} - \frac{1}{(3n+1)(3n+4)}\right]$$

$$= \frac{1}{6}\left[\frac{1}{4} - \frac{1}{(3n+1)(3n+4)}\right]$$

Example # 40 : Find the general term and sum of n terms of the series

$1 + 5 + 19 + 49 + 101 + 181 + 295 + \ ....$

Solution : The sequence of difference between successive term 4, 14, 30, 52, 80 .....

The sequence of the second order difference is 10, 16, 22, 28, ... clearly it is an A.P>

so let nth term

$T_n = an^3 + bn^2 + cn + d$

$a + b + c + d = 1$ ....(i)

$8a + 4b + 2c + d = 5$ ....(ii)

$27a + 9b + 3c + d = 19$ ....(iii)

$64a + 16b + 4c + d = 49$ ....(iv)

from (i), (ii), (iii) & (iv)

$a = 1, b = -1, c = 0, d = 1 \quad \Rightarrow \quad T_n = n^3 - n^2 + 1$

$$s_n = \Sigma(n^3 - n^2 + 1) = \left(\frac{n(n+1)}{2}\right)^2 - \frac{n(n+1)(2n+1)}{6} + n = \frac{n(n^2-1)(3n+2)}{12} + n$$

Self practice problems :

(25) Sum to n terms the following series

(i) $\frac{3}{1^2.2^2} + \frac{5}{2^2.3^2} + \frac{7}{3^2.4^2} + ...$

(ii) $1 + (1 + 2) + (1 + 2 + 3) + (1 + 2 + 3 + 4)$ ....

(iii) $4 + 14 + 30 + 52 + 82 + 114 + ...$

(26) If $\sum_{r=1}^{n} T_r = (n + 1)(n + 2)(n + 3)$ then find $\sum_{r=1}^{n} \frac{1}{T_r}$

Ans. (25) (i) $\frac{2n+n^2}{(n+1)^2}$ (ii) $\frac{n(n+1)(n+2)}{6}$ (iii) $n(n + 1)^2$ (26) $\frac{n}{6(n+2)}$