High Level Problems (HLP)

1. Prove that :

(i) $\sec^4 A\,(1-\sin^4 A) - 2\tan^2 A = 1$ (ii) $\dfrac{\cot^2\theta\ (\sec\theta-1)}{1+\sin\theta} = \sec^2\theta\ \dfrac{1-\sin\theta}{1+\sec\theta}$.

2. Simplify the expression $\sqrt{\sin^4 x + 4\cos^2 x} - \sqrt{\cos^4 x + 4\sin^2 x}$

3. Let a, b, c, d be numbers in the interval $[0, \pi]$ such that $\sin a + 7\sin b = 4(\sin c + 2\sin d)$, $\cos a + 7\cos b = 4(\cos c + 2\cos d)$. Prove that $2\cos(a-d) = 7\cos(b-c)$.

4. Prove that $(4\cos^2 9^\circ - 3)(4\cos^2 27^\circ - 3) = \tan 9^\circ$

5. If $\cos(\alpha+\beta) = \frac{4}{5}$; $\sin(\alpha-\beta) = \frac{5}{13}$ & α, β lie between 0 & $\frac{\pi}{4}$, then find the value of $\tan 2\alpha$.

6. If α & β are two distinct roots of the equation $a\tan\theta + b\sec\theta = c$, then prove that $\tan(\alpha+\beta) = \dfrac{2ac}{a^2 - c^2}$.

7. If $\tan\alpha = \frac{p}{q}$ where $\alpha = 6\beta$, α being an acute angle, prove that;

$\frac{1}{2}(p\,\text{cosec}\,2\beta - q\sec 2\beta) = \sqrt{p^2 + q^2}$.

8. If $\sin(\theta+\alpha) = a$ & $\sin(\theta+\beta) = b$ $(0 < \alpha, \beta, \theta < \pi/2)$ then find the value of $\cos 2(\alpha-\beta) - 4ab\cos(\alpha-\beta)$

9. Show that:

(i) $\cot 7\frac{1^\circ}{2}$ or $\tan 82\frac{1^\circ}{2} = (\sqrt{3}+\sqrt{2})(\sqrt{2}+1)$ or $\sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$

(ii) $\tan 142\frac{1^\circ}{2} = 2 + \sqrt{2} - \sqrt{3} - \sqrt{6}$.

10. If $\tan\beta = \dfrac{\tan\alpha + \tan\gamma}{1 + \tan\alpha\,.\,\tan\gamma}$, prove that $\sin 2\beta = \dfrac{\sin 2\alpha + \sin 2\gamma}{1 + \sin 2\alpha\,.\sin 2\gamma}$.

11. If α & β satisfy the equation $a\cos 2\theta + b\sin 2\theta = c$ then prove that: $\cos^2\alpha + \cos^2\beta = \dfrac{a^2 + ac + b^2}{a^2 + b^2}$.

12. Show that: $4\sin 27^\circ = (5+\sqrt{5})^{1/2} - (3-\sqrt{5})^{1/2}$

13. If $xy + yz + xz = 1$, then prove that

$$\frac{x}{1-x^2} + \frac{y}{1-y^2} + \frac{z}{1-z^2} = \frac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}.$$

14. Let $a = \frac{\pi}{7}$

(a) Show that $\sin^2 3a - \sin^2 a = \sin 2a \sin 3a$
(b) Show that $\text{cosec}\, a = \text{cosec}\, 2a + \text{cosec}\, 4a$
(c) Evaluate $\cos a - \cos 2a + \cos 3a$
(d) Prove that $\cos a$ is a root of the equation $8x^3 - 4x^2 - 4x + 1 = 0$
(e) Evaluate $\tan a \tan 2a \tan 3a$
(f) Evaluate $\tan^2 a + \tan^2 2a + \tan^2 3a$
(g) Evaluate $\tan^2 a \tan^2 2a + \tan^2 2a \tan^2 3a + \tan^2 3a \tan^2 a$
(h) Evaluate $\cot^2 a + \cot^2 2a + \cot^2 3a$

15. In a ΔABC, prove that $\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2} = 1+4\sin\left(\frac{\pi-A}{4}\right)\sin\left(\frac{\pi-B}{4}\right)\sin\left(\frac{\pi-C}{4}\right)$

16. Evaluate

$\cos a \cos 2a \cos 3a \ldots\ldots\ldots \cos 999a$, where $a = \frac{2\pi}{1999}$

17. Prove that the average of the numbers
$2\sin 2^\circ, 4\sin 4^\circ, 6\sin 6^\circ, \ldots\ldots\ldots 180\sin 180^\circ$ is $\cot 1^\circ$

18. Solve $\tan 2\theta = \tan\frac{2}{\theta}$.

19. Find the general values of x and y satisfying the equations
$5\sin x\cos y = 1$, $4\tan x = \tan y$

20. Solve $\dfrac{\sin^3\frac{x}{2}-\cos^3\frac{x}{2}}{2+\sin x} = \dfrac{\cos x}{3}$.

21. Solve the system of equations :
$x + y = \frac{2\pi}{3}$, $\sin x + \sin y = \frac{3}{2}$ and $x, y \in \left[0, \frac{\pi}{2}\right]$

22. Solve the following system of simultaneous equations for x and y:
$4^{\sin x} + 3^{1/\cos y} = 11$
$5.16^{\sin x} - 2.3^{1/\cos y} = 2$

23. Solve $\cos\theta + \sin\theta = \cos 2\theta + \sin 2\theta$.

24. Solve $8\sin x = \dfrac{\sqrt{3}}{\cos x} + \dfrac{1}{\sin x}$

25. Solve the equation $\sin^3 x\cos 3x + \cos^3 x\sin 3x + 0.375 = 0$.

26. Solve the equation $\frac{\sqrt{3}}{2}\sin x - \cos x = \cos^2 x$.

27. Solve the equation $\sin^4 x + \cos^4 x - 2\sin^2 x + \frac{3}{4}\sin^2 2x = 0$

28. Solve for x, the equation $\sqrt{13-18\tan x} = 6\tan x - 3$, where $-2\pi < x < 2\pi$.

29. Solve the equation $3 - 2\cos\theta - 4\sin\theta - \cos 2\theta + \sin 2\theta = 0$

30. Solve the equation $\sin^2 4x + \cos^2 x = 2\sin 4x \,.\, \cos^4 x$

31. Prove that : $\cos 5A = 16\cos^5 A - 20\cos^3 A + 5\cos A$

32. If $\cos\theta = \frac{1}{2}\left(a+\frac{1}{a}\right)$ and $\cos 3\theta = \frac{1}{2}\left(a^k+\frac{1}{a^k}\right)$ then number of natural numbers 'k' less than 50 is (given $a \in R$)

33. Consider the equation for $0 \le \theta \le 2\pi$; $\left(\sin 2\theta + \sqrt{3}\cos 2\theta\right)^2 - 5 = \cos\left(\frac{\pi}{6} - 2\theta\right)$. If greatest value of θ is $\frac{k\pi}{p}$ (k, p are coprime), then find the value of (k + p).

Answers

2. $\cos^2 x - \sin^2 x = \cos 2x$

5. $\frac{56}{33}$

8. $1 - 2a^2 - 2b^2$

14. (c) $\frac{1}{2}$ (e) $\sqrt{7}$ (f) 21 (g) 35 (h) 5

16. $\frac{1}{2^{999}}$

18. $\frac{n\pi}{4} \pm \sqrt{1+\frac{n^2\pi^2}{16}}$, $n \in I$

20. $x = (4n + 1)\frac{\pi}{2}, n \in I$

21. $x = \frac{\pi}{2}$, $y = \frac{\pi}{6}$ or $x = \frac{\pi}{6}, y = \frac{\pi}{2}$.

22. $x = n\pi + (-1)^n \frac{\pi}{6}, y = 2n\pi \pm \frac{\pi}{3}$

23. $2n\pi, n \in I$ or $\frac{2n\pi}{3} + \frac{\pi}{6}, n \in I$

24. $x = n\pi + \frac{\pi}{6}, n \in I, x = \frac{n\pi}{2} - \frac{\pi}{12}, n \in I$

25. $x = \frac{n\pi}{4} + (-1)^{n+1} . \frac{\pi}{24}; n \in I$

26. $x = (2n + 1)\pi : n \in I, x = 2n\pi + \frac{\pi}{3}, n \in I$

27. $x = n\pi \pm \frac{1}{2}\cos^{-1}\left(2-\sqrt{5}\right), n \in I$

28. $\alpha - 2\pi;\ \alpha - \pi,\ \alpha,\ \alpha + \pi$, where $\tan \alpha = \frac{2}{3}$

29. $\theta = (4n + 1)\pi/2, \theta = 2n\pi, n \in I$

30. $x = (2n + 1)\frac{\pi}{2}, n \in I$

32. 25

33. 31